Indag. Mathem., N.S., 20 (1), 87–100 March, 2009

Generalized balancing numbers ☆

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Communicated by Prof. R. Tijdeman

ABSTRACT

The positive integer x is a (k,l)-balancing number for y $(x \leqslant y-2)$ if

$$1^k + 2^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l,$$

for fixed positive integers k and l. In this paper, we prove some effective and ineffective finiteness statements for the balancing numbers, using certain Baker-type Diophantine results and Bilu–Tichy theorem, respectively.

1. INTRODUCTION

Let y, k and l be fixed positive integers with $y \geqslant 4$. We call the positive integer x $(\leqslant y-2)$ a (k,l)-*power numerical center for* y, or a (k,l)-*balancing number for* y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l. \tag{1}$$

☆ K. Liptai was supported by the Hungarian National Foundation for Scientific Research Grant No. T 048945 MAT. F. Luca was supported by Grant SEP-CONACyT 46755 and a Guggenheim Fellowship. Á. Pintér was supported by the Hungarian National Foundation for Scientific Research Grants Nos. K 75566, K 067580 and by the Bolyai Scholarship. L. Szalay was supported by the Hungarian National Foundation for Scientific Research Grant No. T 048945 and by the Eötvös Scholarship of Hungary.

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The special case $k = l$ of this definition is due to Finkelstein [9] who proved that infinitely many integers y possess $(1, 1)$-power centers (see also [2]), and that there is no integer $y > 1$ with a $(2, 2)$-power numerical center. The proofs depend on the theory of Pell equations and the resolution of the Thue equations $X^3 + 2Y^3 = 11$ and 33, respectively, in integers X, Y. We note that the particular case $k = l = 1$ is strongly related to another problem called the house problem (see, for example, [1]). In his paper, Finkelstein conjectured that if $k > 1$, then there is no integer $y > 1$ with a (k, k)-power numerical center. Later, using a result of Ljunggren [11] and Cassels [7] on triangular numbers whose squares are also triangular, Finkelstein [14] confirmed his own conjecture for $k = 3$. Recently, Ingram [10] proved Finkelstein's conjecture for $k = 5$.

In this paper, we prove a general result about (k, l)-balancing numbers. Unfortunately, we cannot deal with Finkelstein's conjecture in its full generality. However, we obtain the following theorem.

Theorem 1. *For any fixed positive integer $k > 1$, there are only finitely many positive pairs of integers (y, l) such that y possesses a (k, l)-power numerical center.*

The case $k = l$ has already been dealt with recently by Ingram [10] with a method similar to ours. The proof splits naturally in two cases. The first case is when $1 \leqslant l \leqslant k$. Since k is fixed and there are only finitely any such l, we may assume that l is also fixed. Our proof now uses an ineffective statement of Rakaczki [13]. The second case is when $k < l$, and here we show by Runge's method that there are no positive integers y possessing a (k, l) numerical center.

Because of the use of the result from [13], our Theorem 1 is ineffective in case $l \leqslant k$ in the sense that we cannot provide an upper bound for possible numerical centers x in terms of k unless $l = 1$ or $l = 3$. In these cases, we have the following theorem.

Theorem 2. *Let k be a fixed positive integer with $k \geqslant 1$ and $l \in \{1, 3\}$. If $(k, l) \neq (1, 1)$, then there are only finitely many (k, l)-balancing numbers, and these balancing numbers are bounded by an effectively computable constant depending only on k.*

We note that some numerical centers do exist. For example, in the case $(k, l) = (2, 1)$, we can rewrite equation (1) as

$$2x^3 + 4x = 3y^2 - 3y,$$

which is an elliptic curve whose short Weierstrass normal form is $u^3 + 72u + 81 = v^2$ (via the bi-rational transformation $v = 18y - 9$ and $u = 6x$). Using the program package MAGMA, we solved this elliptic equation and its solutions lead to three $(2, 1)$-balancing numbers x, namely 5, 13 and 36.

2. AUXILIARY RESULTS

For the proofs of our results, we need some auxiliary results. For an integer $k \geqslant 1$, we write

$$S_k(x) = 1^k + 2^k + \cdots + (x-1)^k.$$

These expressions are strongly related to the *Bernoulli polynomials*. In the next lemma, we summarize some of the well-known properties of Bernoulli polynomials. For the proofs, we refer to [12].

Lemma 1. *Let $B_n(X)$ denote the nth Bernoulli polynomial and put $B_n = B_n(0)$ for $n = 1, 2, \ldots$. Further, let D_n be the denominator of B_n. We then have:*

(A) $S_k(X) = \frac{1}{k+1}(B_{k+1}(X) - B_{k+1})$;
(B) $B_n(X) = X^n + \sum_{k=1}^{n} \binom{n}{k} B_k X^{n-k}$;
(C) $B_n(X) = (-1)^n B_n(1-X)$;
(D) $B_1 = -\frac{1}{2}$ *and* $B_{2n+1} = 0$ *for* $n \geqslant 1$;
(E) *(von Staudt–Clausen)* $D_{2n} = \prod_{p-1|2n,\, p \text{ prime}} p$;
(F) 0 *and* 1 *are double zeros of* $S_k(X)$ *for odd values of* $k \geqslant 3$. *Further,* 0 *and* 1 *are simple zeros of* $S_k(X)$ *for even values of* k.

We shall also need the following lemma (for some results of a similar flavor, see the Appendix to [3]).

Lemma 2. *Let p be prime. Assume that the sum of the digits of n in base p is $\geqslant p$. Then there exists an even positive integer $k < n$ such that p divides the denominator of the rational number*

$$\binom{n}{k} B_k$$

when written in reduced form.

Proof. Let first p be odd. Then

$$n = n_1 p^{\alpha_1} + \cdots + n_t p^{\alpha_t},$$

here $0 \leqslant \alpha_1 < \cdots < \alpha_t$ and $n_1, \ldots, n_t \in \{1, \ldots, p-1\}$. We now select non-negative integers m_i for $i = 1, \ldots, t$, such that $m_i \leqslant n_i$ and $\sum_{i=1}^{t} m_i = p - 1$. It is clear that this can be done since $\sum_{i=1}^{t} n_i \geqslant p$. We put

$$k = \sum_{i=1}^{t} m_i p^{\alpha_i}.$$

Then, $k < n$. Further, reducing k modulo $p-1$, we get that $k \equiv \sum_{i=1}^{t} m_i \pmod{p-1}$, therefore $(p-1) \mid k$. In particular, k is even and p divides the denominator of B_k.

Finally, by a well-known theorem of Lucas (see [8], p. 271, items 76 and 77), we have

$$\binom{n}{k} \equiv \prod_{i=1}^{t} \binom{n_i}{m_i} \pmod{p},$$

therefore p does not divide $\binom{n}{k}$, which completes the proof of the lemma in this case.

If $p = 2$, then $n = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_t}$ with $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_t$ and $t \geqslant 2$. Then, taking $k = 2^{\alpha_2}$, we see that $k < n$, k is even, and, by Lucas's Theorem, $\binom{n}{k}$ is odd. Since 2 divides the denominator of B_k, we get that 2 divides the denominator of $\binom{n}{k} B_k$. □

The next lemma is based on a recent deep theorem of Bilu and Tichy [4], as well as on the indecomposability of the Bernoulli polynomials proved by Bilu et al. in [3].

To present Lemma 3, we define special pairs $(l, g(X))$ as follows. In the sequel, we let $\delta(X) \in \mathbb{Q}[X]$ be a linear polynomial, and $q(X) \in \mathbb{Q}[X]$ be a non-zero polynomial. Further, for l odd, $S_l(X)$ can be written in the form $\phi_l((X - 1/2)^2)$ with some appropriate polynomial $\phi_l(X) \in \mathbb{Q}[X]$, see [13]. We now define special pairs $(l, g(X))$ as follows:

Special pair of type I: $(l, S_l(q(X)))$, where $q(X)$ is not constant.

Special pair of type II: l is odd and $g(X) = \phi_l(\delta(X)q(X)^2)$.

Special pair of type III: l is odd and $g(X) = \phi_l(c\delta(X)^t)$, where $c \in \mathbb{Q} \setminus \{0\}$ and $t \geqslant 3$ is an odd integer.

Special pair of type IV: l is odd and $g(X) = \phi_l((a\delta(X)^2 + b)q(X)^2)$, where $a, b \in \mathbb{Q} \setminus \{0\}$.

Special pair of type V: l is odd and $g(X) = \phi_l(q(X)^2)$.

Special pair of type VI: $l = 3$ and $g(X) = \delta(X)q(X)^2$.

Special pair of type VII: $l = 3$ and $g(X) = q(X)^2$.

Lemma 3. *Let l be a positive integer and $g(X) \in \mathbb{Q}[X]$ be a polynomial of degree greater than 2. Then the equation*

$$S_l(x) = g(y)$$

has only finitely many integer solutions x and y, unless $(l, g(X))$ is a special pair.

Proof. This is Theorem 1 in [13]. □

We now rewrite equation (1) using the polynomials $S_k(x)$ and $S_l(y)$ as the Diophantine equation

$$S_k(x) + S_l(x+1) = S_l(y). \tag{2}$$

Lemma 4. *Assume that $k < l$ and put*

(3) $$P(X) = (X+1)^l - S_k(X) \in \mathbb{Q}[X].$$

Put x_0 for the largest real root of $P(X)$. If x and y is an integer solution to the Diophantine equation (2)*, then $x \leqslant x_0$. In particular, if either $P(X)$ has no real root, or $x_0 < 2$, then the Diophantine equation* (2) *has no integer solutions $x \geqslant 2$ and $y \geqslant x+2$.*

Proof. Suppose that the integers $x \geqslant 2$ and $y \geqslant x+2$ satisfy (2). Since $k < l$, it follows that the leading coefficient of $P(X)$ is positive and $\deg(P) = l$.

Clearly, if $P(X)$ does not possess a real root, or if $x_0 < 2$, or if $2 \leqslant x_0 < x$, then $P(x) = (x+1)^l - S_k(x) > 0$. Then

(4) $$-x^l - S_k(x) < 0 < (x+1)^l - S_k(x).$$

Increasing both sides of the inequality (4) by $x^l + S_k(x) + S_l(x)$, we get

(5) $$S_l(x) < x^l + S_k(x) + S_l(x) < (x+1)^l + x^l + S_l(x),$$

which leads to

(6) $$S_l(x) < S_k(x) + S_l(x+1) < S_l(x+2).$$

Since (x, y) is a integer solution of (2), we get that $S_k(x) + S_l(x+1)$ can be replaced in (6) by $S_l(y)$. Thus,

(7) $$S_l(x) < S_l(y) < S_l(x+2).$$

The properties of the polynomial S_l together with inequalities (7) imply that $y = x+1$. Thus, by (2), $S_k(x) = 0$, which is impossible. □

The following three results yield information on the structure of zeros of certain polynomials.

Lemma 5. *Let $p(X) = a_n X^n + \cdots + a_1 X + a_0$ be a polynomial with integral coefficients for which a_0 is odd, $4 \mid a_i$ for all $i = 1, \ldots, n$, and $\mathrm{ord}_2(a_n) = 3$. Then every zero of p is simple.*

Proof. This is Lemma 4 in [6]. □

One of the most surprising theorem on zeros of certain polynomials related to $S_k(x)$ is due to Voorhoeve, Győry and Tijdeman [15]. They proved if $k \notin \{1, 3, 5\}$ then the polynomial $S_k(x) + R(x)$ possesses at least three zeros of odd multiplicities for every polynomial $R(x)$ with rational *integer* coefficients. Of course, this general statement is not true for some polynomials with rational coefficients, however, we obtain Lemmas 6 and 7.

Lemma 6. *Each one of the polynomials*

$$F_1(X) = 8S_k(X) + (2X+1)^2, \quad k > 1,$$

has at least three zeros of odd multiplicities.

Proof. Let d be the smallest positive integer for which

$$8d\big(B_{k+1}(X) - B_{k+1}\big) \in \mathbb{Z}[X].$$

Then, by (D) and (E) of Lemma 1, d is an odd square-free integer. We now show that the polynomial

$$P_1(X) = 8d\big(B_{k+1}(X) - B_{k+1}\big) + d(k+1)(2X+1)^2$$

has at least three zeros of odd multiplicities. Note that the leading coefficient of P_1 equals $8d$. Since d is odd and is the smallest positive integer such that $8d(B_{k+1}(X) - B_{k+1}) \in \mathbb{Z}[X]$, it follows that the content of $P_1(X)$ (i.e. the greatest common divisor of all its coefficients) is a power of 2 dividing 8.

If k is even, the fact that $P_1(X)$ has at least three simple zeros is a simple consequence of Lemma 5.

Assume now that k is odd. If $P_1(X)$ is associated to a complete square in $\mathbb{Q}[X]$, then we get that

$$P_1(X) = aR(X)^2, \tag{8}$$

where a is an integer and $R(X) \in \mathbb{Z}[X]$ is a polynomial with positive leading term. By writing $a = a_1r^2$, with integers a_1 and r such that a_1 is square-free, and by replacing $R(X)$ by $rR(X)$, we may assume that a is square-free. It is clear that $a > 0$. We also assume that $k \geqslant 5$, since the case $k = 3$ can be checked by hand.

If $2 \parallel k+1$, it is then easy to see that the content of $P_1(X)$ is 2 (all the coefficients of $P_1(X)$ are even, and the last is $d(k+1)$, therefore it is not a multiple of 4). Hence, by Gauss Lemma, $a = 2$ and the content of $R(X)$ is 1. Writing

$$R(X) = a_0X^{(k+1)/2} + a_1X^{(k-1)/2} + \cdots + a_{(k+1)/2},$$

and identifying the first three coefficients in (8), we get

$$8d = aa_0^2, \qquad -4d(k+1) = 2aa_0a_1, \qquad \frac{2dk(k+1)}{3} = a\big(a_1^2 + 2a_0a_2\big).$$

The first relation above forces $a_0 = 2$ and $d = 1$. The third relation above shows that a_1 is even, therefore $2aa_0a_1$ is a multiple of 16. Now the second relation above contradicts the fact that $2 \parallel k+1$.

If $4 \mid k+1$, then the content of $P_1(X)$ is 4, unless (see Lemma 2) $k+1$ is a power of 2, in which case it is 8. Thus, $a = 1$, unless $k+1$ is a power of 2, in which case $a = 2$. Identifying leading terms in (8), we get that $8d = aa_0^2$, therefore $d = 1$

and $a=2$. Thus, $k+1=2^\alpha$, for some $\alpha \geqslant 3$. However, since $d=1$, it follows, by Lemma 2, that the sum of the digits of $k+1$ in base 3 is at most 2. Thus, since $k+1$ is not a power of 3 (because $k+1$ is even), we get that $k+1=3^\beta+3^\gamma$ for some $0 \leqslant \beta \leqslant \gamma$. Hence, $2^\alpha=3^\beta+3^\gamma$, therefore $\beta=0$. Since the largest solution of the Diophantine equation $2^\alpha=1+3^\gamma$ is $\alpha=2$, $\gamma=1$, we get $k+1=4$, therefore $k=3$, which is a contradiction.

We now have to exclude the remaining case in which

$$P_1(X)=\big(aX^2+bX+c\big)R^2(X), \tag{9}$$

where both aX^2+bX+c and $R(X)$ are in $\mathbb{Z}[X]$, such that aX^2+bX+c has two distinct zeros. Up to replacing $R(X)$ by $-R(X)$, we may assume that the leading coefficient of $R(X)$ is positive. Further, because the content of $P_1(X)$ is a power of 2 dividing 8, it follows that $\gcd(a,b,c)$ is a power of 2. By writing $\gcd(a,b,c)=2^\alpha$ for some non-negative integer α, and replacing $R(X)$ by $2^{\lfloor\alpha/2\rfloor}R(X)$, we may assume that $\alpha=0$, or 1. Hence, $\gcd(a,b,c)$ is either 1 or 2.

If $k+1$ is even but not divisible by 4, then $P_1(X)/2$ is a polynomial in $\mathbb{Z}[X]$ having odd constant term and all other coefficients even. Thus, $P_1(X)/2 \equiv 1 \pmod 2$. Hence, it can be factored as

$$P_1(X)/2=\big(2S_1(X)+1\big)^2\big(2S_2(X)+1\big),$$

with some polynomials $S_i(X)\in\mathbb{Z}[X]$ for $i=1,2$. However, the leading coefficient of $P_1(X)/2$ is not divisible by 8.

Assume now that $4 \mid k+1$. When $k=3$, one can check by hand that $P_1(X)$ does not have the form shown at (9). Assume now that $k+1\geqslant 8$. Note that the content of $P_1(X)$ is 4, unless $k+1$ is power of 2, when it is 8. It now follows that $R_1(X)=R(X)/2\in\mathbb{Z}[X]$. Thus,

$$P_1(X)/4=\big(aX^2+bX+c\big)R_1(X)^2. \tag{10}$$

We now write

$$R_1(X)=a_0X^{(k-1)/2}+a_1X^{(k-1)/2-1}+a_2X^{(k-1)/2-2}+\cdots+a_{(k-1)/2}.$$

Identifying the first three coefficients in $P_1(X)/4$, we get, on the one hand the polynomial

$$P_1(X)/4=2dX^{k+1}-d(k+1)X^k+\frac{dk(k+1)}{6}X^{k-1}+\cdots,$$

while on the other hand the polynomial

$$\begin{aligned}&\big(aX^2+bX+c\big)\big(a_0^2X^{k-1}+2a_0a_1X^{k-2}+\big(a_1^2+2a_0a_2\big)X^{k-3}+\cdots\big)\\&\quad=aa_0^2X^{k+1}+\big(ba_0^2+2aa_0a_1\big)X^k\\&\qquad+\big(ca_0^2+2ba_0a_1+a\big(a_1^2+2a_0a_2\big)\big)X^{k-1}+\cdots\end{aligned}$$

which leads to the relations

$$aa_0^2 = 2d, \qquad ba_0^2 + 2aa_0a_1 = -d(k+1)$$

and

$$ca_0^2 + 2ba_0a_1 + a\big(a_1^2 + 2a_0a_2\big) = \frac{dk(k+1)}{6}.$$

The first relation above shows that $a_0 = 1$, $a = 2d$. The second one now becomes

(11) $\qquad b = -d(k+1+4a_1),$

while the third one now reads

(12) $\qquad c = d\big(2(k+1)a_1 + 6a_1^2 - 4a_2\big) + \dfrac{d(k(k+1)}{6}.$

Clearly, $d \mid a$ and the above relations (11) and (12) for b and c, show that $d \mid b$, and if there exists a prime $p > 3$ such that $p \mid d$, then $p \mid \gcd(a, b, c)$. Since $\gcd(a, b, c) \in \{1, 2\}$, we get that $d = 1$ or $d = 3$. To rule out the possibility that $d = 3$, assume that $3 \mid k+2$. Then the above formula (12) for c shows that $d = 3$. Identifying the last coefficient in (9) and using the fact that $S_k(0) = 0$ (by (F) of Lemma 1), we get

(13) $\qquad d(k+1) = c(2a_{(k-1)/2})^2,$

and since 3 divides d but not $k+1$, we get that $3 \mid c$. Since $d \mid a$ and $d \mid b$, we obtain $3 \mid \gcd(a, b, c)$, which is again a contradiction. Thus, 3 does not divide $k+2$, therefore $3 \mid k(k+1)$. Now relation (12), shows again that $d \mid c$. Hence, $d \mid \gcd(a, b, c)$, which implies that $d = 1$.

Lemma 2 implies now that the sum of the digits of $k+1$ is base 3 is $\leqslant 2$. Since $4 \mid k+1$, we get that $k+1$ cannot be a power of 3, therefore $k+1 = 3^{\alpha_0} + 3^{\alpha_1}$ for some $0 \leqslant \alpha_0 \leqslant \alpha_1$. Since $4 \mid k+1$, and $k+1 = 3^{\alpha_0}(3^{\alpha_1-\alpha_0}+1)$, we get that $4 \mid 3^{\alpha_1-\alpha_0}+1$. This shows that $\alpha_1 - \alpha_0$ is odd. Further, since for an odd positive integer s, we have that $4 \parallel 3^s + 1$, we get that $(k+1)/4$ is odd. Clearly, $2 \parallel a$ and relation (12) show that c is even. Now relation (13) together with the facts that c is even and $4 \parallel k+1$ leads to a contradiction. $\square$

Lemma 7. *Each one of the polynomials*

$$F_2(X) = 4S_k(X) + X^2(X+1)^2, \quad k \geqslant 1, k \neq 3,$$

has at least three zeros of odd multiplicities.

Proof. We follow the same method as in the proof of Lemma 6. We use again d for the least positive integer such that $4d(B_{k+1}(X) - B_{k+1}) \in \mathbb{Z}[X]$. By (D) and (E) of Lemma 1, we have that d is odd and square-free.

For the polynomial $F_2(X)$, we have

$$P_2(X) = 4d\big(B_{k+1}(X) - B_{k+1}\big) + d(k+1)X^2(X+1)^2 \in \mathbb{Z}[X].$$

Assume first that $k+1$ is odd. We then have to exclude the case when

$$P_2(X) = (aX+b)R^2(X),$$

where $aX+b$ and $R(X) \in \mathbb{Z}[X]$. Since d is square-free and odd, further 0 is a simple zero of $P_2(X)$ (by (F) of Lemma 1), we obtain

$$P_2(X) = aXR^2(X).$$

The coefficient of X in $R^2(X)$ is even, thus the coefficient of X^2 in $P_2(X)$ is also even, which is a contradiction.

Assume now that $k+1 > 4$ is even. Then, we have to exclude that either $P_2(X) = aR(X)^2$, or $P_2(X) = (aX^2+bX+c)R(X)^2$ with some polynomial $R(X) \in \mathbb{Z}[X]$, and some integers a, b and c, with $a \neq 0$.

We first look at the case $P_2(X) = aR(X)^2$. We may assume again that $R(X)$ has positive leading coefficient and that $a > 0$ is square-free. The content of $P_2(X)$ is a power of 2, therefore $a = 1$ or 2. We assume that $k+1 > 8$, since the smaller cases can be checked by hand. Writing

$$R(X) = a_0X^{(k+1)/2} + a_1X^{(k+1)/2-1} + \cdots$$

and identifying the first 5 coefficients we get, on the one hand, that $P_2(X)$ is the polynomial

$$\begin{aligned}&4dX^{k+1} - 2d(k+1)X^k + \frac{d(k+1)k}{3}X^{k-1}\\ &\quad - \frac{d(k+1)k(k-1)(k-2)}{180}X^{k-3}\end{aligned}$$

(note that $k-3 > 4$, therefore the first 5 terms in $P_2(X)$ are the same as the first 5 terms in $4dS_k(X)$), while on the other hand, we get the polynomial

$$\begin{aligned}&aa_0^2X^{k+1} + 2aa_0a_1X^k + a\big(a_1^2 + 2a_0a_2\big)X^{k-1} + a(2a_0a_3 + 2a_1a_2)X^{k-2}\\ &\quad + a\big(a_2^2 + 2a_0a_4 + 2a_1a_3\big)X^{k-3} + \cdots.\end{aligned}$$

Hence, we obtain the relations

$$(14) \qquad 4d = aa_0^2, \qquad -2d(k+1) = 2aa_0a_1, \qquad \frac{d(k+1)k}{3} = aa_1^2 + 2aa_0a_2,$$

as well as

$$(15) \qquad 2aa_0a_3 + 2aa_1a_2 = 0 \quad \text{and} \quad aa_2^2 + 2aa_0a_4 + 2aa_1a_3 = -\binom{k+1}{4}\frac{2d}{15}.$$

The first relation in (14) above together with $a \in \{1,2\}$, gives $d = 1$, $a_0 = 2$, and $a = 1$. Hence, by Lemma 2, $k+1$ is either a power of 3, or of the form $3^\alpha + 3^\beta$ for some $0 \leqslant \alpha \leqslant \beta$. Since $k+1$ is even, it cannot be a power of 3, therefore $k+1 = 3^\alpha(3^{\beta-\alpha}+1)$. If $\beta - \alpha$ is even, then $2 \parallel k+1$, and if $\beta-\alpha$ is odd, then $4 \parallel k+1$. Now, the second relation in (14) above gives $a_1 = -(k+1)/2$, and the third relation in (14) above together with the fact that $k+1$ is even shows that a_1 is even. Thus, $4 \parallel k+1$, which shows that $\binom{k+1}{4}$ is odd. Finally, since a_0 and a_1 are both even, reducing the second of relations (15) modulo 4 we get

$$a_2^2 \equiv 2 \pmod 4,$$

which is the desired contradiction in this case.

It remains to deal with the case when

$$P_2(X) = \bigl(aX^2 + bX + c\bigr)R(X)^2, \tag{16}$$

where a, b, c are integers and $R(X) \in \mathbb{Z}[X]$. As before, we assume that $R(X)$ has positive leading coefficient, that $a > 0$, and $\gcd(a,b,c) = 1$ or 2.

We write

$$R(X) = a_0 X^{(k-1)/2} + a_1 X^{(k-1)/2-1} + \cdots + a_{(k-1)/2},$$

assume that $k \geqslant 5$ and identify the first three coefficients in (16) to get

$$aa_0^2 = 4d, \qquad ba_0^2 + 2a_0aa_1 = -2d(k+1), \tag{17}$$

and

$$ca_0^2 + 2ba_0a_1 + a\bigl(a_1^2 + 2a_0a_2\bigr) = \frac{dk(k+1)}{3}. \tag{18}$$

The first relation (17) shows that $d \mid a$ and then the second relation (17) shows that $d \mid b$. Now relation (18) shows that if there exists a prime $p > 3$ dividing d, then $p \mid \gcd(a,b,c)$, which is a contradiction. Thus, d is a divisor of 3. To rule out the case $d = 3$, suppose that $3 \mid k+2$. Then, relation (18) shows that $d = 3$. Evaluating relation (16) at $x = 1$ and using (F) of Lemma 1 (for $S_k(1) = 0$), we get

$$4d(k+1) = (a+b+c)R(1)^2.$$

Since $3 \mid k+2$, we get that 3 does not divide $k+1$. Hence, $3 \mid (a+b+c)$, and since 3 divides both a and b, we get that it divides also c, which is again a contradiction. Thus, $d = 1$, and since $k+1$ is even, we get that $k+1 = 3^\alpha + 3^\beta$ with $0 \leqslant \alpha \leqslant \beta$. If $\beta - \alpha$ is even, then $2 \parallel k+1$ and if $\beta - \alpha$ is odd, then $4 \parallel k+1$.

If $4 \parallel k+1$, then $4\binom{k+1}{4}B_4$ is the coefficient of X^{k-3} in $P_2(X)$ and it is even but not a multiple of 4, while if $2 \parallel k+1$, then $4\binom{k+1}{2}B_2$ is the coefficient of X^{k-1} in $P_2(X)$, and is also even but not a multiple of 4. In conclusion, the content of $P_2(X)$ is 2, which means that $\gcd(a,b,c) = 2$.

We now return to relations (17) and (18) and note that $a_0 \in \{1,2\}$. If $a_0 = 2$, then $a = 1$, which is false. Thus, $a_0 = 1$ and $a = 4$. Now the second relation (17) shows that $4 \mid b$. Relation (18) shows that $4 \mid c$ too, which leads to the contradiction $4 = \gcd(a,b,c)$, unless $2 \parallel k+1$. So, let us assume that $2 \parallel k+1$. Note that in this case relation (18) implies that $c \equiv 2 \pmod 4$.

Now let us note that by (F) of Lemma 1, we have that 0 is a double roots of $S_k(X)$. Thus, 0 is also double root of $P_2(X)$. Hence, $a_{(k-1)/2} = 0$. Put $F(X) = P_2(X)/X^2$, and $R_1(X) = R(X)/X$. Identifying the *last* two coefficients in the equation

$$\begin{aligned} F(X) &= \bigl(aX^2 + bX + c\bigr)R_1(X)^2 \\ &= \bigl(aX^2 + bX + c\bigr)\bigl(\cdots + 2a_{(k-3)/2}a_{(k-5)/2}X + a_{(k-3)/2}^2\bigr), \end{aligned}$$

we get

$$\begin{aligned} &2k(k+1)B_{k-1} + k + 1 = c(a_{(k-3)/2})^2, \\ &2(k+1) = 2ca_{(k-3)/2}a_{(k-5)/2} + ba_{(k-3)/2}^2. \end{aligned} \tag{19}$$

The first relation (19) shows that $2k(k+1)B_{k-1} \in \mathbb{Z}$. Since B_{k-1} is a rational number whose denominator is an even square-free integer, it follows that $2k(k+1)B_k$ is congruent to 2 modulo 4. Since $2 \parallel k+1$, we get that the left-hand side of the first equation (19) is a multiple of 4. Since $2 \parallel c$, we get that $a_{(k-3)/2}$ is even. We now immediately get that the right-hand side of the second equation (19) is a multiple of 8, whereas its left-hand side is not. This final contradiction concludes the proof of this lemma. □

Finally, we recall a special case of a result by Brindza [5].

Lemma 8. *Let $f(X) \in \mathbb{Q}[X]$ be a polynomial having at least three zeros of odd multiplicities. Then the equation*

$$f(x) = y^2$$

in integers x and y implies that $\max(|x|,|y|) < c$, where c is an effectively computable constant depending only on the coefficients and degree of f.

Proof. See Theorem in [5]. □

3. PROOFS

We start with the proof of Theorem 2 since it will be needed in the proof of Theorem 1.

Proof of Theorem 2. Since

$$S_1(x) = \frac{x(x-1)}{2} \quad \text{and} \quad S_3(x) = \left(\frac{x(x-1)}{2}\right)^2,$$

equation (1) leads to the equations

$$8S_k(x) + (2x+1)^2 = (2y-1)^2$$

and

$$4S_k(x) + \big(x(x+1)\big)^2 = \big(y(y-1)\big)^2,$$

respectively. Now, apart from the case when in the second equation $k = 3$, the fact that such equations have only finitely many effectively computable integer solutions is a simple consequence of Lemmas 6–8. We recall that Finkelstein resolved the case $(k,l) = (3,3)$. Thus, our proof is complete. □

Proof of Theorem 1. We first assume that $k = l > 3$.

If x is a (k,k)-balancing number for y, then, from (1), we have

$$2S_k(x) + x^k = S_k(y).$$

Recall that for odd values of k, $S_k(X) = \phi_k((X - 1/2)^2)$ with an appropriate polynomial $\phi_k(X)$ with rational coefficients, and the leading coefficient of $S_k(X)$ and $\phi_k(X)$ is $1/(k+1)$. We show that the identities

$$2S_k(X) + X^k = S_k\big(\delta(X)\big)$$

and, for odd k,

$$2S_k(X) + X^k = \phi_k\big(q(X)\big),$$

where $\delta(X)$ and $q(X)$ are polynomials with rational coefficients of degree 1 and 2, respectively, are impossible. Indeed, if the leading coefficient of $\delta(X)$ or $q(X)$ is $a \in \mathbb{Q}$, then the leading coefficient of $S_k(\delta(X))$ or $\phi_k(q(X))$ is $a^{k+1}/(k+1)$ or $a^{\frac{k+1}{2}}/(k+1)$, respectively, which cannot be $2/(k+1)$. Thus, $(k, 2S_k(X) + X^k)$ is not a standard pair, and now Lemma 3 completes the proof.

We follow a similar approach for $k > l \geqslant 1$. By Theorem 2, we may assume that $l \notin \{1,3\}$.

Since k is fixed, and there are only finitely many such l, we may assume that l is also fixed. Then, from (2), we have

$$S_k(x) + S_l(x+1) = S_l(y).$$

We prove that the identities

$$S_k(X) + S_l(X+1) = S_l\big(q(X)\big) \tag{20}$$

and, for odd l,

$$S_k(X) + S_l(X+1) = \phi_l\big(q(X)\big), \tag{21}$$

where $q(X)$ is a polynomial with rational coefficients, are impossible. Let d and a be the degree and leading coefficient of $q(X)$. On comparing the degrees and the leading coefficients in (20) and (21), we obtain

$$k+1=(l+1)d \quad \text{and} \quad \frac{1}{k+1}=\frac{a^d}{l+1},$$

and

$$k+1=\frac{l+1}{2}d \quad \text{and} \quad \frac{1}{k+1}=\frac{a^d}{l+1}, \tag{22}$$

respectively. From these relations, we get $a^d=1/d$ or $2/d$. Since $d>1$, we see that $a=1/b$, where b is an integer, and $b^d=d$ or $b^d=d/2$. The first equation has no integer solutions with $d>1$, while the only solution in integers of the second equation with $d>1$ is $d=2, b=\pm 1$. However, when $d=2$, the first relation (22) shows that $k=l$, which is not allowed.

We now deal with the case $l>k$.

In this case, the fact that equation (2) has no positive integer solutions x and y is a direct consequence of Lemma 4. By that lemma, it is sufficient to show that the polynomial $P(X)=(X+1)^l-S_k(X)$ has no real zero $\geqslant 2$. The estimate

$$S_k(x)=1^k+2^k+\cdots+(x-1)^k<\int_0^x t^k\,dt=\frac{x^{k+1}}{k+1}\leqslant\frac{x^l}{2},$$

provides

$$P(x)=(x+1)^l-S_k(x)>(x+1)^l-\frac{x^l}{2}>0$$

for all $x\geqslant 0$. Thus, there is no (k,l)-balancing number with $k<l$. □

ACKNOWLEDGEMENTS

The authors are grateful to the referee for his/her remarks and suggestions.

REFERENCES

[1] Adams J.P. – Puzzles for Everybody, Avon Publications, New York, 1955, p. 27.
[2] Behera A., Panda G. – On the square roots of triangular numbers, Fibonacci Quart. **37** (1999) 98–105.
[3] Bilu Yu., Brindza B., Kirschenhofer P., Pintér Á., Tichy R. – Diophantine equations and Bernoulli polynomials (with an appendix by Schinzel, A.), Compos. Math. **131** (2002) 173–188.
[4] Bilu Yu., Tichy R. – The diophantine equation $f(x)=g(y)$, Acta Arith. **95** (2000) 261–288.
[5] Brindza B. – On S-integral solutions of the equations $y^m=f(x)$, Acta Math. Hungar. **44** (1984) 133–139.
[6] Brindza B., Pintér Á. – On equal values of power sums, Acta Arith. **77** (1996) 97–101.
[7] Cassels J.W. – Integral points on certain elliptic curves, Proc. London Math. Soc. **3** (1965) 55–57.
[8] Dickson L.E. – History of the Theory of Numbers, vol. I, New York, Chelsea, 1952.
[9] Finkelstein R.P. – The House problem, Amer. Math. Monthly **72** (1965) 1082–1088.

[10] Ingram P. – On k-th power numerical centres, C. R. Math. Acad. Sci. R. Can. **27** (2005) 105–110.
[11] Ljunggren W. – Solution complète de quelques équations du sixième degré à deux indéterminées, Arch. Math. Naturv. **48** (1946) 177–211.
[12] Rademacher H. – Analytic Number Theory, Springer-Verlag, New York–Heidelberg, 1973.
[13] Rakaczki Cs. – On the diophantine equation $S_m(x) = g(y)$, Publ. Math. Debrecen **65** (2004) 439–460.
[14] Steiner R. – On kth-power numerical centers, Fibonacci Quart. **16** (1978) 470–471.
[15] Voorhoeve M., Győry K., Tijdeman R. – On the Diophantine equation $1^k + 2^k + \cdots + x^k + R(x) = y^z$, Acta Math. **143** (1979) 1–8.

(Received November 2007)