INTRODUCTION TO FRACTAL GEOMETRY

*R. Horváth** - *L. Németh** - *L. Szalay** - *J. Závoti***

Preface

Simply speaking, fractal geometry is a mathematical description of self-similar objects, and fractals are shapes which show similar features at different scales. Most people have probably seen complex and often beautiful images known as fractals. Many books (see e.g. Peitgen and Richter (1986), Peitgen and Saupe (1988), Peitgen et al. (1991) and Szabó (1997)) have been written on the subject, but they were written primarily for high-level studies in mathematics. The purpose of this text is to show how some fractals are generated, and how they can be applied to geosciences.

Koch Curve and Koch Snowflake

Koch curve is named after Helge von Koch, a Swedish mathematician, who invented it. There is a simple method to construct the Koch curve. Suppose that an arbitrary straight line segment is given as the initial object. The process of the construction consists of three steps.

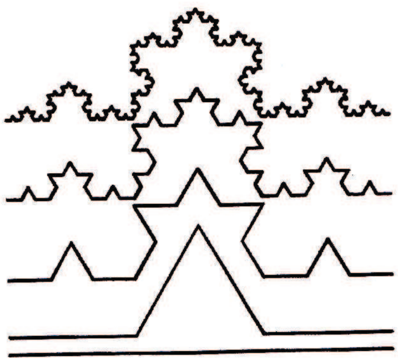

Fig. 1. Stages of the Koch curve.

Step 1. Divide the segment into three equal parts and remove the middle section.
Step 2. Replace the missing section with an equilateral triangle and take away its base.
Step 3. Take each of the getting line segments, and apply the Process to all of them.

Some first stages of this construction can be seen in Fig. 1. The Koch curve (or the Koch fractal) is the limit of the approximating stages, which has interesting features. For instance, it has no unique tangent, it has infinite length because the nth stage consists of 4^n segments with length $\frac{1}{3^n}$, so the total length $\left(\frac{4}{3}\right)^n$ tends to infinity (as n tends to infinity). Other feature is the self-similarity, i.e. we can partition the Koch curve into 4 equivalent parts and each part is similar to the whole curve. It is not too difficult to produce self-similar objects, a few examples are in the Fig. 2.

*University of Sopron, Institute of Mathematics
rhorvath@efe.hu, lnemeth@efe.hu, lszalay@efe.hu
**GGRI of HAS, Sopron
zavoti@ggki.hu

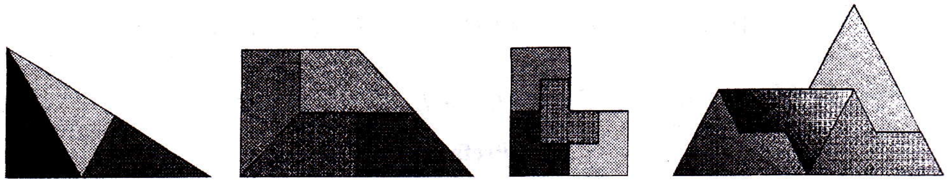

Fig. 2. Self-similar plane figures.

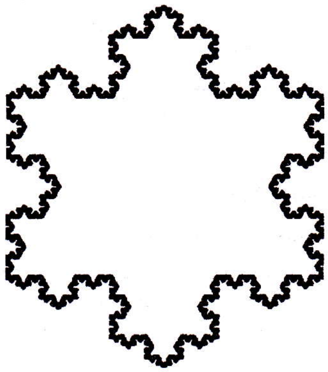

Fig. 3. The Koch snowflake.

Instead of a straight line segment, apply the steps of the process to an initial equilateral triangle. This infinite construction leads to a new object called Koch snowflake or Koch island (see Fig. 3.). Because the length of the Koch curve is infinite, the perimeter of the Koch snowflake is obviously infinite. But what about the area?

Let a be the side of the initial triangle, then its area is $A_0 = \frac{\sqrt{3}}{4}a^2$. It is easy to see that the growth of the area is $A_n = A_0 \frac{1}{3}\left(\frac{4}{3^2}\right)^{n-1}$ if we step across from the $(n-1)$th iteration to the nth one. Using the well-known formula for the sum of geometric series, the total area of the snowflake is $\sum_{k=0}^{\infty} A_k = \frac{2\sqrt{3}}{5}a^2$. This result is very incredible! Why? Because infinite perimeter encloses finite area! This phenomenon cannot be described by classical geometry. We need something new to do it, and this new idea is fractal geometry.

At the end of this section, without completeness, we enumerate some famous and historical fractals as keywords of orientation: Cantor set (or Cantor Dust, see Fig. 4.), Sierpinski gasket (or Sierpinski triangle, see Fig. 5.), Sierpinski carpet, Sierpinski tetrahedron, Menger sponge, Peano-Hilbert curves (see Fig. 6.), Devil's staircase, Mandelbrot set, Julia set.

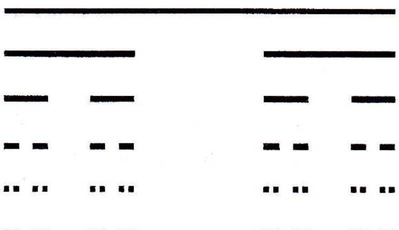

Fig. 4.
Approximations of the Cantor set

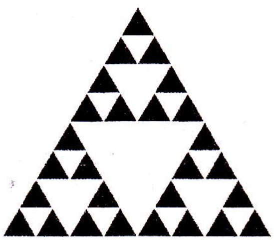

Fig. 5.
A stage of the Sierpinski gasket

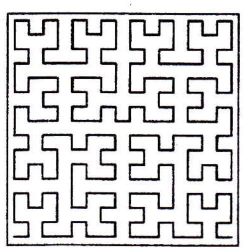

Fig. 6.
Peano-Hilbert curves

Fractals and Scale, Self-Similarity Dimension

What shall we do if we want to measure the length of a coast line? Take a stick and put it down sometimes end-to-end, near the line. The result is coming soon, if we multiply the length of the stick L by the number of the sticks N (see Fig. 7.). The problem is that the length of the coast line depends on the length of stick.

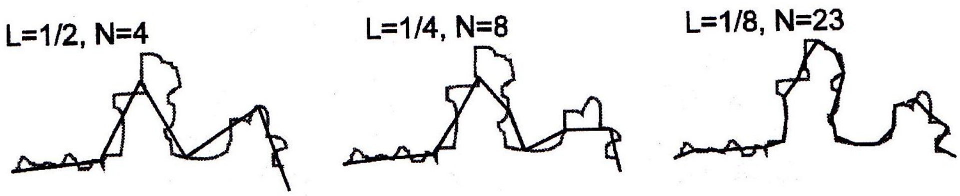

Fig. 7. Measurements of the coast line.

This problem has been studied by Mandelbrot (1967). Mandelbrot mentioned a story about the length of the common border between Portugal and Spain. The length of this border was found about 150 miles more in Portugal than in Spain, supposedly they used different sticks for measurement. Mandelbrot's famous paper – „How long is the coast of Britain?" – also deals with the above theme. To solve this problem Mandelbrot introduced the idea of an object which is independent of the scale. He called it fractal derived from the Latin word „fractus". The property of the independence of the scale means self-similarity we have already written about. He also proposed the expression „fractal dimension", based on the self-similarity feature. Before defining it, remember some simple facts. Lines are considered to have one dimension, surfaces have two dimensions, and solids have three dimensions. If we take a straight line segment, to double its length we need 2 copies of the original segment. Taking a square, we need $4 = 2^2$ copies to double its length and widths. In case of a cube, for doubling the sides, $8 = 2^3$ copies are necessary.

The dimension of the object is the exponent. Let us think this fact over concerning the Koch curve. To get a triple sized Koch fractal we need $4 = 3^{\log_3 4}$ copies, which means that the dimension is $\log_3 4$! It is very surprising, because the result is not a positive integer. The self-similarity dimension of self-similar fractals is defined the following manner:

$$D = \frac{\log n}{\log k},$$

where n is the number of the pieces to get k multiple enlargement of the object. The following table gives the self-similarity dimension of fractals mentioned earlier.

Fractal	Self-similarity dimension
Koch curve	log 4 / log 3
Sierpinski gasket (Fig. 8.)	log 3 / log 2
Cantor set	log 2 / log 3

Other notions of dimension – for example box-counting dimension, topological dimension, Hausdorff-Besitkovich dimension – are also introduced and used in science.

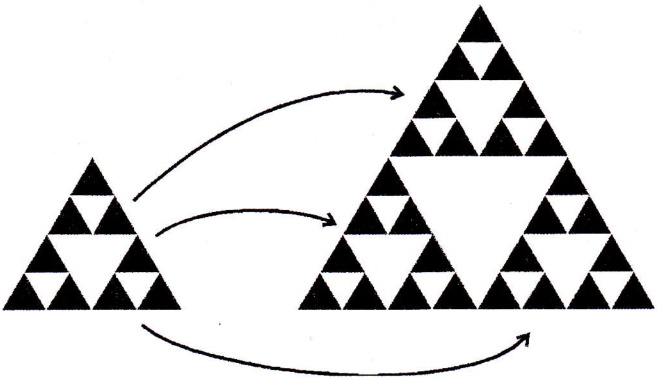

Fig. 8. The self-similarity dimension of the Sierpinski gasket is $\frac{\log 3}{\log 2}$.

What are Fractals?

Many mathematicians have given definitions of fractal, but all definitions were retracted or have not been found satisfactory. Falconer (1990) suggested, that without definition, recognise fractals by their properties, for instance self-similarity, fine structure, cannot be described by classical geometry, etc. Biologists do the same with the expression „life". So there has not existed a good definition yet.

Fractals and Geoscience

In the end we would like to call the readers' attention that there exist books, articles, conferences concerning the application of fractals to geoscience. Turcotte's (1992) book deals with this topic. K. Musgrave's remarkable article on „Building Fractal Planets" can be reached on the following INTERNET World Wide Web address: http://www.seas.gwu.edu/faculty/musgrave/ article.html/. The 3rd International Symposium on Fractals and Dynamic Systems in Geoscience was held in Slovakia in 1997. We have just mentioned a few examples to indicate the importance of fractals in geoscience, and hope that the above article aroused the reader's interest in studying fractals.

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