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Balancing Diophantine triples with distance 1

Murat Alp · **Nurettin Irmak** · **László Szalay**

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Abstract For a positive real number w let the Balancing distance $\|w\|_B$ be the distance from w to the closest Balancing number. The Balancing sequence is defined by the initial values $B_0 = 0$, $B_1 = 1$ and by the binary recurrence relation $B_{n+2} = 6B_{n+1} - B_n$, $n \geq 0$. In this paper, we show that there exist only one positive integer triple (a, b, c) such that the Balancing distances $\|ab\|_B$, $\|ac\|_B$ and $\|bc\|_B$ all are exactly 1.

Keywords Diophantine triples · Balancing numbers

Mathematics Subject Classification 11D72 · 11B39

1 Introduction

Let $\{B_n\}_{n\geq 0}$ denote the sequence of Balancing numbers given by $B_0 = 0$, $B_1 = 1$ and $B_{n+2} = 6B_{n+1} - B_n$ for all integers $n \geq 0$. For a positive real number w we define the Balancing distance of w by

$$\|w\|_B = \min\{|w - B_n| \ : \ n \geq 0\}.$$

This notion is analogous to the Fibonacci distance $\|w\|_F$ introduced by Luca et al. [1]. They showed that if $1 \leq a < b < c$ are integers, then

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} > \exp(0.034\sqrt{\log c}).$$

M. Alp · N. Irmak
Mathematics Department, Art and Science Faculty, Niğde University, Niğde 51100, Turkey
e-mail: muratalp@nigde.edu.tr

N. Irmak
e-mail: nirmak@nigde.edu.tr

L. Szalay (✉)
Institute of Mathematics, University of West Hungary, Ady E. út 5, Sopron 9400, Hungary
e-mail: szalay.laszlo@emk.nyme.hu

This result has a numerical corollary, namely if

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} \le 2, \tag{1.1}$$

then $c \le \exp(415.62)$. In fact, the solution with maximal c to inequality (1.1) is

$$(a, b, c) = (1, 11, 235).$$

Note, that the origin of the problem is to solve the system of Diophantine equations $ab + 1 = G_x$, $ac + 1 = G_y$ and $bc + 1 = G_z$, where the sequence $\{G\}_{n=0}^{\infty}$ satisfies a given recurrence relation of order two. For more information, see [1–4]. The main result of this work is the following.

Theorem 1.1 *Suppose that ε_x, ε_y and ε_z are all in the set $\{\pm 1\}$. If there exist integers $1 \le a < b < c$ such that*

$$\begin{aligned} ab + \varepsilon_x &= B_x, \\ ac + \varepsilon_y &= B_y, \\ bc + \varepsilon_z &= B_z \end{aligned} \tag{1.2}$$

hold with positive integers x, y and z, then

$$(a, b, c) = (1, 34, 1188), \quad (x, y, z) = (3, 5, 7), \quad (\varepsilon_x, \varepsilon_y, \varepsilon_z) = (1, 1, -1). \tag{1.3}$$

The associate sequence of $\{B_n\}$ is denoted by $\{C_n\}$. It is known that $C_0 = 2$, $C_1 = 6$, and $C_{n+2} = 6C_{n-1} - C_n$ $(n \ge 0)$, moreover

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad C_n = \alpha^n + \beta^n, \tag{1.4}$$

where $\alpha = 3 + 2\sqrt{2}$, $\beta = 3 - 2\sqrt{2}$. These explicit formulae (or the common recurrence relation of $\{B_n\}$ and $\{C_n\}$) make it possible to define the Balancing sequence and its associate sequence for negative subscripts, too.

In the next section we collect the auxiliary results we need in the proof of Theorem 1.1.

2 Preliminary results

Lemma 2.1 *Any non-negative integer n satisfies*

1. $B_{2n} \equiv 0 \pmod 6$,
2. $B_{4n+\varepsilon} \equiv \varepsilon \pmod 6$, *where* $\varepsilon \in \{\pm 1\}$,
3. $C_n \equiv 2 \pmod 4$.

Proof The statements immediately come from the fact that the sequences $\{B_n\}$ and $\{C_n\}$ are periodic for any modulus. Considering the initial values, together with the common recurrence relation, the proof of the lemma is an easy consequence. □

Lemma 2.2 *Assume that $n \ge m$ are arbitrary non-negative integers. Then the following identities hold.*

1. $B_{n+m} + B_{n-m} = B_n C_m$,
2. $B_{n+m} - B_{n-m} = C_n B_m$,
3. $B_{n+m} B_{n-m} = (B_n + B_m)(B_n - B_m)$,

4. $B_{2n} + \varepsilon = (B_n - \varepsilon B_{n-1})(B_{n+1} + \varepsilon B_n)$, *where* $\varepsilon \in \{\pm 1\}$.

Proof All formulae can be proved by using (1.4). We remark that the first three identities have already been appeared in [6], but this work is relatively inaccessible. Identity (3) was also shown in [7].

Here we deal only with (4). Since $\alpha\beta = 1$, expanding the product $(B_n - \varepsilon B_{n-1})(B_{n+1} + \varepsilon B_n)$ we find

$$\frac{(\alpha-\beta)\alpha^{2n} - (\alpha-\beta)\beta^{2n} + (\alpha-\beta)^2\varepsilon}{(\alpha-\beta)^2} = B_{2n} + \varepsilon.$$

□

Sometimes it facilitates the usage of Lemma 2.2, if one specifies the parameters as follows.

Corollary 2.3 1. $B_{n+1} + B_{n-1} = 6B_n$ *and* $B_{n+1} - B_{n-1} = C_n$ *follow from Lemma* 2.2 *(1) and (2), respectively, by taking* $m = 1$,
2. $B_{2n-1} + 1 = B_n C_{n-1}$ *and* $B_{2n-1} - 1 = B_{n-1} C_n$ *can be deduced from Lemma* 2.2 *(1) and (2), respectively, by* $m = n - 1$,
3. $B_{2n} = B_n C_n$ *is a consequence of case* $n = m$ *of Lemma* 2.2 *(1) (or (2)),*
4. $B_{2n-1} = (B_n + B_{n-1})(B_n - B_{n-1})$ *is implied by the case* $n = m - 1$ *of Lemma* 2.2 *(3).*

Lemma 2.4 *Suppose that* n *and* m *are arbitrary non-negative integers, at least one of them is positive, further put* $\nu = \gcd(n, m)$. *Then*

1. $\gcd(B_n, B_m) = B_\nu$,
2. $\gcd(C_n, C_m) = \begin{cases} C_\nu, & \textit{if } \frac{n}{\nu} \equiv \frac{m}{\nu} \equiv 1 \pmod 2; \\ 2, & \textit{otherwise,} \end{cases}$
3. $\gcd(B_n, C_m) = \begin{cases} C_\nu, & \textit{if } \frac{n}{\nu} \not\equiv \frac{m}{\nu} \equiv 1 \pmod 2; \\ 1 \textit{ or } 2, & \textit{otherwise,} \end{cases}$
4. $\gcd(B_{n+3} + \varepsilon B_{n+2}, B_{n+1} + \varepsilon B_n) = 1$, *where* $\varepsilon \in \{\pm 1\}$,
5. $\gcd(B_{n+2} + \varepsilon B_{n+1}, B_{n+1} + \varepsilon B_n) = 1$, *where* $\varepsilon \in \{\pm 1\}$.

Proof The first three results are well-known from Carmichael's general work [8].

To show the next to last property, observe that the terms $s_n = B_n + \varepsilon B_{n-1}$ of the sequence $\{s_n\}$ satisfy $s_n = 6s_{n-1} - s_{n-2}$, $s_0 = -\varepsilon$, $s_1 = 1$. Clearly, $2 \nmid s_n$ and $3 \nmid s_n$. If $p \mid \gcd(s_n, s_{n-2})$ holds for some prime $p \geq 5$, then $p \mid s_n + s_{n-2} = 6s_{n-1}$, and then $p \mid s_{n-1}$ follows. Thus p divides three consecutive terms in $\{s_n\}$, consequently $p \mid s_0$, a contradiction.

The treatment of the last statement is similar. □

Lemma 2.5 *For any positive integer* n,

$$\gcd(B_{2n-2} \pm 1, B_n \pm 1) \leq 41615.$$

Proof Put $Q_0 = \gcd(B_{2n-2} \pm 1, B_n \pm 1)$. Applying Lemma 2.2 (3) with $m = 1$ and Lemma 2.4 (1), one can immediately conclude

$$\begin{aligned} Q_0 &\leq \gcd\left(B_{2n-1}B_{2n-3}, B_{n+1}B_{n-1}\right) \\ &\leq \gcd\left(B_{2n-1}, B_{n+1}\right)\gcd\left(B_{2n-1}, B_{n-1}\right) \\ &\quad \times \gcd\left(B_{2n-3}, B_{n+1}\right)\gcd\left(B_{2n-3}, B_{n-1}\right) \\ &\leq B_3 \cdot B_1 \cdot B_5 \cdot B_1 = 41615. \end{aligned}$$

□

Lemma 2.6 *Let $n \geq 2$ denote a positive integer, and let $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. For the greatest common divisor $Q_1 = gcd(B_n - \varepsilon_1, B_{n-2} - \varepsilon_2)$,*

1. $Q_1 \leq 68$ *if* $\varepsilon_1 = \varepsilon_2$;
2. $Q_1 = \left\{ \begin{array}{ll} B_k - B_{k-1}, & \text{if } n = 2k; \\ 6C_{2k+1}, & \text{if } n = 4k+3; \\ C_{2k}, & \text{if } n = 4k+1 \end{array} \right\}$ *if* $\varepsilon_1 = -1$ *and* $\varepsilon_2 = 1$;
3. $Q_1 = \left\{ \begin{array}{ll} B_k + B_{k-1}, & \text{if } n = 2k; \\ 2B_{2k+1}, & \text{if } n = 4k+3; \\ 6B_{2k}, & \text{if } n = 4k+1 \end{array} \right\}$ *if* $\varepsilon_1 = 1$ *and* $\varepsilon_2 = -1$.

hold.

Proof (1) Applying Lemma 2.2 (3), the defining relation of the sequence $\{B_n\}$ and Lemma 2.4 (1), one has

$$\begin{aligned} Q_1 &= \gcd(B_n - \varepsilon_1, B_n - B_{n-2}) \leq \gcd(B_{n+1}B_{n-1}, B_n - B_{n-2}) \\ &\leq \gcd(B_{n+1}, 2B_n - 6B_{n-1}) \cdot \gcd(B_{n-1}, 6B_{n-1} - 2B_{n-2}) \\ &= \gcd(B_{n+1}, 6B_{n+1} - 34B_n) \cdot \gcd(B_{n-1}, 2B_{n-2}) \leq 34 \cdot 2. \end{aligned}$$

(2) Here the principal tool is, according to the parity of n, the application of Lemma 2.2 (4) and Corollary 2.3 (2).

If $n = 2k$, then

$$B_{2k} + 1 = (B_k - B_{k-1})(B_{k+1} + B_k)$$

and

$$B_{2k-2} - 1 = (B_{k-1} + B_{k-2})(B_k - B_{k-1}).$$

Obviously, the greatest common divisor is $B_k - B_{k-1}$ if and only if $\gcd(B_{k+1} + B_k, B_{k-1} + B_{k-2}) = 1$. But it is clear by Lemma 2.4 (4).

Assume that $n = 4k + 3$. Then Corollary 2.3 (2) shows

$$B_{4k+3} + 1 = B_{2k+2}C_{2k+1} \qquad \text{and} \qquad B_{4k+1} - 1 = B_{2k}C_{2k+1}.$$

Recall Lemma 2.1 (1) to show $Q_1 = 6C_{2k+1}$. Indeed, it is easy to see that $B_{2k+2}/6$ and $B_{2k}/6$ coprimes. (For instance, define the sequence $b_n = B_{2n}/6$. It has initial values $b_0 = 0, b_1 = 1$ and $b_n = 34b_{n-1} - b_{n-2}$ holds for $n \geq 2$. The consecutive terms of $\{b_n\}$ are coprime.)

Suppose $n = 4k + 1$. Now Corollary 2.3 (2) gives

$$B_{4k+1} + 1 = B_{2k+1}C_{2k} \qquad \text{and} \qquad B_{4k-1} - 1 = B_{2k-1}C_{2k},$$

and the statement is a direct consequence of Lemma 2.4 (1) and Lemma 2.1 (2).

(3) We leave to the reader the proof since the analogous way to the proof of Lemma 2.6 (2) works. □

Lemma 2.7 *Let $n \geq 2$ denote a positive integer, and let $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. For the greatest common divisor $Q_2 = gcd(B_{2n-3} - \varepsilon_1, B_n - \varepsilon_2)$, we have*

1. $Q_2 \leq 2380$ *if* $\varepsilon_1 = 1$;
2. $Q_2 = \left\{ \begin{array}{ll} B_k + B_{k-1}, & \text{if } n = 2k; \\ 2B_{2k+1}, & \text{if } n = 4k+3; \\ 6\mu B_{2k}, & \text{if } n = 4k+1 \end{array} \right\}$ *if* $\varepsilon_1 = -1$ *and* $\varepsilon_2 = 1$, *where* $\mu = 1$ *holds unless* $k = 3s + 1$, $\mu = 33$;

3. $Q_2 = \left\{\begin{array}{ll} B_k - B_{k-1}, & \textit{if } n = 2k; \\ 6\mu C_{2k+1}, & \textit{if } n = 4k+3 \\ C_{2k}, & \textit{if } n = 4k+1; \end{array}\right\}$ *if* $\varepsilon_1 = -1$ *and* $\varepsilon_2 = -1$, *where* $\mu = 1$ *holds unless* $k = 3s+2$, $\mu = 33$.

Proof (1) Corollary 2.3 (2), Lemma 2.2 (3), further Lemma 2.4 (1) and (3) imply

$$\begin{aligned} Q_2 &\le \gcd(B_{n-2}C_{n-1}, B_{n+1}B_{n-1}) \\ &\le \gcd(B_{n-2}, B_{n+1}) \gcd(B_{n-2}, B_{n-1}) \\ &\quad \times \gcd(C_{n-1}, B_{n+1}) \gcd(C_{n-1}, B_{n-1}) \\ &\le B_3 \cdot B_1 \cdot C_2 \cdot 2 = 2380. \end{aligned}$$

(2) If $n = 2k$, then according to Corollary 2.3 (2), (4) and Lemma 2.2 (4),

$$B_{4k-3} + 1 = (B_k + B_{k-1})(B_k - B_{k-1})C_{2k-2}$$

and

$$B_{2k} - 1 = (B_k + B_{k-1})(B_{k+1} - B_k).$$

Lemma 2.4 (5) provides $\gcd(B_{k+1} - B_k, B_k - B_{k-1}) = 1$, moreover by Corollary 2.3 (4) we know that $\gcd(B_{k+1} - B_k, C_{2k-2}) \le \gcd(B_{2k+1}, C_{2k-2})$. But the latest greatest common divisor is 1, according to Lemma 2.4 (3) and Lemma 2.1 (2).

If $n = 4k+3$, then consider the following decompositions (via Corollary 2.3 (2) and (3)):

$$B_{8k+3} + 1 = B_{2k+1}C_{2k+1}C_{4k+1} \quad \text{and} \quad B_{4k+3} - 1 = B_{2k+1}C_{2k+2}.$$

Clearly, $\gcd(C_{2k+2}, C_{2k+1}) = 2$ and $\gcd(C_{2k+2}, C_{4k+1}) = 2$ follow in the virtue of Lemma 2.4 (2), which together with Lemma 2.1 (3) show the statement.

Suppose $n = 4k+1$. Now we have

$$B_{8k-1} + 1 = B_{2k}C_{2k}C_{4k-1} \quad \text{and} \quad B_{4k+1} - 1 = B_{2k}C_{2k+1}.$$

It is easy to see, that $\gcd(C_{2k+1}, C_{2k}) = 2$, further $\gcd(C_{2k+1}, C_{4k-1}) = 198$ holds if $3 \mid 2k+1$, otherwise $\gcd(C_{2k+1}, C_{4k-1}) = 6$. Then apply Lemma 2.1 (3) again.

(3) The proof is analogous to the proof of Lemma 2.7 (2), the details are left to the reader. □

Lemma 2.8 *If* $n \ge 3$ *is a natural number, then*

$$\alpha^{n-0.9831} < B_n < \alpha^{n-0.983} \qquad \text{and} \qquad \alpha^n < C_n < \alpha^{n+0.004}.$$

Proof The first statement was shown in [2] (see Lemma 4). Note that from the proof it is clear that $B_n < \alpha^{n-0.983}$ is longer available if $0 \le n \le 2$. The second part follows from a more general result (see Lemma 2.2 in [9]). □

Lemma 2.9 *System* (1.2)*, together with* $2 \le y < z$ *implies* $z \le 2y - 1$.

Proof By system (1.2) and its assertions, further by Lemma 2.8 and $y \ge 2$ we obtain

$$ac = B_y - \varepsilon_y \le B_y + 1 < \alpha^{y-0.983} + 1 < \alpha^{y-0.895}.$$

On the other hand, $c \ge 3$ and $z \ge 3$ admit

$$c > \sqrt{(c-1)c+1} \ge \sqrt{B_z} > \alpha^{(z-0.9831)/2}.$$

Comparing the two estimates, first $z < 2y - 0.8069$ and then $z \le 2y - 1$ follows. □

Lemma 2.10 *If* $z = y + 2 \geq 139$*, then system (1.2) has no solution.*

Proof The proof is split into three parts according to ε_z and ε_y.

I. $\varepsilon_z = \varepsilon_y$. By the virtue of Lemma 2.6 (1), one can conclude

$$c \mid \gcd(B_z - \varepsilon_z, B_y - \varepsilon_z) \leq 68 < \alpha^{2.4}. \tag{2.1}$$

Then we arrived at a contradiction by $z < 6$, which is resulted from $c > \sqrt{B_z} > \alpha^{(z-0.9831)/2}$ and (2.1).

II. $\varepsilon_z = -1$, $\varepsilon_y = 1$. Assume first that z is even, and put $z = 2k$. By Lemma 2.6 (2) and Lemma 2.8 we obtain

$$c \mid \gcd(B_z + 1, B_{z-2} - 1) = B_k - B_{k-1} < B_k < \alpha^{k-0,983}. \tag{2.2}$$

On the other hand, $c > \sqrt{B_z} > \alpha^{(2k-0.9831)/2}$, a contradiction regarding to (2.2).

Much more complicated the case z odd with $z = 4k+3$. Note that now $k \geq 34$. Although, combining Lemma 2.6 (2) and Lemma 2.8

$$c \mid \gcd(B_z + 1, B_{z-2} - 1) = 6C_{2k+1} < \alpha^{1.017+(2k+1.004)} = \alpha^{2k+2.021}$$

follow, the argument $c > \sqrt{B_z} > \alpha^{(4k+2.0169)/2}$ leads no to the desired contradiction. Let d denote a positive integer for which $cd = 6C_{2k+1}$. Thus

$$d = \frac{6C_{2k+1}}{c} < \frac{\alpha^{2k+2.021}}{\alpha^{2k+1.00845}} = \alpha^{1.01255} < 5.96,$$

so $d \in \{1, 2, 3, 4, 5\}$. Hence $c = 6C_{2k+1}/d$, and system (1.2) together with Corollary 2.3 (2) gives $a = B_{2k}/6$ and $b = B_{2k+2}/6$. Obviously, we need only to solve the equation

$$\left(\frac{d}{6}\right)^2 B_{2k}B_{2k+2} + \varepsilon_x = B_x. \tag{2.3}$$

First, it is easy to conclude from the estimates of Lemma 2.8 and (2.3) that $B_{4k-2} < B_x < B_{4k+1}$ hold. Then the case $x = 4k$, via Lemma 2.2 (4), leads to

$$d^2 B_{2k+2} = 36\left(\varepsilon_x B_{2k} + B_{2k+1} - B_{2k-1} - \varepsilon_x \frac{B_{2k+1}B_{2k-1}}{B_{2k}}\right).$$

Consequently, $36B_{2k+1}B_{2k-1}/B_{2k}$ is an integer, which together with Lemma 2.4 (1) contradicts $k \geq 34$. If $x = 4k - 1$, then using the factorization appears in Corollary 2.3 (2), either

$$d^2 B_{2k+2} = 36C_{2k-1} \qquad \text{or} \qquad d^2 B_{2k}B_{2k+2} = 36B_{2k+1}C_{2k}.$$

Since $d \leq 5$ and $\gcd(B_{2k+2}, C_{2k-1})$ is small the first possibility is excluded. Similarly, by $\gcd(B_{2k+1}, B_{(2k+1)\pm 1}) = 1$ the second instance is out of the question, too.

To complete case II, finally we assume $z = 4k + 1$. Consider

$$c \mid \gcd(B_z + 1, B_{z-2} - 1) = C_{2k} < \alpha^{2k+0.004}$$

and $c > \sqrt{B_z} > \alpha^{(4k+0.0169)/2}$, which lead to a contradiction by

$$d = \frac{C_{2k}}{c} < \frac{\alpha^{2k+0.004}}{\alpha^{2k+0.00845}} < 1.$$

III. $\varepsilon_z = 1$, $\varepsilon_y = -1$. Suppose first that $z = 2k$. By Lemma 2.6 (3) and Lemma 2.8, we obtain

$$c \mid \gcd(B_z + 1, B_{z-2} - 1) = B_k + B_{k-1} < \alpha^{k-0,983} + \alpha^{k-1.983}. \tag{2.4}$$

Moreover, $c > \sqrt{B_z} > \alpha^{(2k-0.9831)/2}$ holds, which contradicts (2.4).

Similar conclusion comes from $z = 4k + 3$, since

$$c \mid \gcd(B_z + 1, B_{z-2} - 1) = 2B_{2k+1} < \alpha^{2k-0.417}$$

and $c > \sqrt{B_z} > \alpha^{2k+1}$.

Now let $z = 4k + 1$. Then

$$\alpha^{(4k+0.0169)/2} < \sqrt{B_z} < c \mid \gcd(B_z + 1, B_{z-2} - 1) = 6B_{2k} < \alpha^{2k+0.034}.$$

Clearly, there exists a positive integer d such that $cd = 6B_{2k}$. Now

$$d = \frac{6B_{2k}}{c} < \frac{\alpha^{2k+0.034}}{\alpha^{2k+0.00845}} < \alpha^{0.03} < 1.1$$

imply $d = 1$. Thus $c = 6B_{2k}$, further $a = C_{2k-1}/6$ and $b = C_{2k+1}/6$ follow from Corollary 2.3 (2). Hence we must solve the Diophantine equation

$$C_{2k-1}C_{2k+1} + 36\varepsilon_x = 36B_x.$$

Applying Lemma 2.8, since k is large enough, and so x, we have

$$\alpha^{x+1.04} < 36B_x = C_{2k-1}C_{2k+1} + 36 < \alpha^{4k+0.009},$$

and similarly

$$\alpha^{x+1.05} > 36B_x \geq C_{2k-1}C_{2k+1} - 36 > \alpha^{4k-0.001}.$$

Combining the results above, $x = 4k - 1$ follows. Then either

$$C_{2k+1} = 36B_{2k} \qquad \text{or} \qquad C_{2k-1}C_{2k+1} = 36B_{2k-1}C_{2k}.$$

Both of them are impossible since k is large enough (consider $\gcd(B_{2k}, C_{2k-1})$ and $\gcd(C_{2k}, C_{(2k)\pm 1})$, respectively). $\square$

Lemma 2.11 *If $z = 2y - 3 \geq 139$, then there is no solution to system* (1.2).

Proof Note that the condition $z \geq 139$ yields $y \geq 71$.

I. $\varepsilon_z = 1$, $\varepsilon_y = \pm 1$. According to Lemma 2.7 (1) and Lemma 2.8 we arrive at a contradiction by

$$\alpha^{(2y-3.9831)/2} < \sqrt{B_{2y-3}} < c \mid \gcd(B_{2y-3} - 1, B_y - \varepsilon_y) \leq 2380 < \alpha^{4.5}.$$

II. $\varepsilon_z = -1$, $\varepsilon_y = \pm 1$. Varying $\varepsilon_y = \pm 1$ and the results of Lemma 2.7 (2) and (3), exactly 8 variations exist. We handle them together since the same idea can be applied in each case.

Observe first that by Lemma 2.7 (2) and (3), $Q = \gcd(B_{2y-3} + 1, B_y - \varepsilon_y)$ does not exceed

$$198B_{\lfloor (y+1)/2 \rfloor} < \alpha^{3.001}\alpha^{(y+1)/2-0.983} < \alpha^{y/2+2.6}.$$

On the other hand, $c \mid Q$ further $c > \alpha^{((2y-3)-0.9831)/2} > \alpha^{y-2}$. Conferring $y - 2$ and $y/2 + 2.6$ we are at a contradiction again. $\square$

3 Proof of the theorem

Suppose that $1 \leq a < b < c$ and $\varepsilon_x, \varepsilon_y, \varepsilon_z \in \{\pm 1\}$ satisfy system (1.2) for some non-negative integer x, y and z. Then $B_x = ab + \varepsilon_x \geq 1 \cdot 2 + \varepsilon_x \geq 1$, and $x \geq 1$. Similarly, $B_y = ac + \varepsilon_y \geq 2$ implies $y \geq 2$. Thus $B_y \geq 6$. However, $y > x$ is not necessary, only $y \geq x$ follows. Indeed, if $a = 1$ or $a = 2$, then $B_y - B_x = a(c-b) + \varepsilon_y - \varepsilon_x \leq 0$ is possible. It is easy to see, that $B_y - B_x \neq -1$, i.e. $y \geq x$. The difference $B_z - B_y = c(b-a) + \varepsilon_z - \varepsilon_y$ is always positive, so $z > y$ holds. Subsequently, we deduce $1 \leq x \leq y < z$ and then $z \geq 3$. These observations will be important when the small cases will be verified by computer.

Now we must distinguish between two situations.

Case 3.1 $z \geq 139$

Put $P = \gcd\left(B_z - \varepsilon_z, B_y - \varepsilon_y\right)$. Taking $B_m = 1$ in Lemma 2.2 (3), together with Lemma 2.4 (1) we have

$$\begin{aligned} P &\leq \gcd\left(B_{z-1}B_{z+1}, B_{y-1}B_{y+1}\right) \\ &\leq \prod_{i,j\in\{\pm 1\}} \gcd\left(B_{z-i}, B_{y-j}\right) = \prod_{i,j\in\{\pm 1\}} B_{\gcd(z-i,y-j)}. \end{aligned} \tag{3.1}$$

Obviously, there exists a positive integer k_{ij} such that $\gcd(z-i, y-j) = (z-i)/k_{ij}$.

Suppose that $k_{ij} \geq 8$ holds for any pair $(i, j) \in \{\pm 1\}^2$. Since $c \mid P$, then Lemma 2.8 implies

$$\alpha^{(z-1)/2} < \sqrt{B_z} < c \leq P \leq B^2_{(z-1)/8} B^2_{(z+1)/8} < \alpha^{4((z+1)/8-0.983)}. \tag{3.2}$$

Comparing the exponents of α in (3.2), we arrive at a contradiction.

Hence $k_{ij} \leq 7$ is necessarily true for at least one possible pair (i, j). Let k denote this k_{ij}. Further assume that

$$\frac{z-i}{k} = \frac{y-j}{\ell}$$

holds for a suitable positive integer ℓ coprime to k.

Suppose for the moment that $\ell > k$. Then $z - i < y - j$ leads to $z = y + 1$ via $y < z$. Thus Lemma 2.2 (3) and Lemma 2.4 (1) show

$$\begin{aligned} P &= \gcd\left(B_{y+1} - \varepsilon_z, B_y - \varepsilon_y\right) \leq \gcd\left(B_{y+2}B_y, B_{y+1}B_{y-1}\right) \\ &= \gcd\left(B_{y+2}, B_{y-1}\right) \leq B_3 = 35 < \alpha^{2.1}. \end{aligned}$$

Hence, by the first part of (3.2), we have a contradiction by $(z-1)/2 < 2.1$.

Assume now that $\ell = k$. Trivially, $k = \ell = 1$. Since $z - i = y - j$, we obtain $z = y + 2$, which provides no solution in the virtue of Lemma 2.10.

In the sequel, we assume $\ell < k$. First analyze the case $2 \leq k/\ell$. Here

$$z = \frac{k}{\ell}(y-j) + i \geq 2(y-1) - 1 = 2y - 3,$$

which, together with Lemma (2.9) implies the following three possibilities: $z = 2y - 3$, $z = 2y - 2$ and $z = 2y - 1$.

(a) If $z = 2y - 3$, then Lemma (2.11) handles the problem.
(b) If $z = 2y - 2$, then hanging together Lemma 2.5, it follows that

$$\alpha^{\frac{z-0.9831}{2}} < \sqrt{B_z} < c \leq \gcd\left(B_{2y-2} - \varepsilon_z, B_y - \varepsilon_y\right) \leq 41615 < \alpha^{5.1},$$

and then $z < 10.3$.

(c) When $z = 2y - 1$ holds, then by Lemma 2.8 we have

$$\frac{b}{a} = \frac{B_{2y-1} - \varepsilon_z}{B_y - \varepsilon_y} \geq \frac{B_{2y-1} - 1}{B_y + 1} > \frac{\alpha^{2y-1-0.9831} - 1}{\alpha^{y-0.983} + 1} > \alpha^{y-1.2}.$$

Subsequently,

$$a^2\alpha^{y-1.2} - 1 < ab - 1 \leq B_x < \alpha^{x-0.983}$$

follow. Since $a^2\alpha^{y-1.2} - 1 \geq a^2(\alpha^{y-1.2} - 1) > a^2\alpha^{y-1.4}$, we can deduce

$$a^2 < \frac{\alpha^{x-0.983}}{\alpha^{y-1.4}} = \alpha^{x-y+0.417}.$$

Obviously, $y > x$ leads to a contradiction by $\alpha^{x-y+0.417} \leq \alpha^{-0.587} < 1$. That is $y = x$ and now $\alpha^{x-y+0.417} = \alpha^{0.417} < 2.1$ entails $a = 1$. This specific case, together with (1.2) easily provides $b = B_y - 1$, $c = B_y + 1$, moreover

$$B_{2y-1} - \varepsilon_z = B_y^2 - 1.$$

Note that $\varepsilon_z = 1$ admits $B_{2y-1} = B_y^2$, which contradicts the Primitive Divisor Theorem (see [10]) since y is large enough. Supposing $\varepsilon_z = -1$, we obtain

$$B_{2y-1} + 1 = B_y^2 - 1.$$

But Corollary 2.3 (2) comes up with $B_{2y-1} + 1 = B_y C_{y-1}$. Hence B_y must divide $B_y^2 - 1$, which is impossible because of $y \geq 2$.

Finally, assume $k/\ell < 2$. Note that it implies $k \geq 3$. Taking any pair $(i_0, j_0) \neq (i, j)$, we have

$$z - i_0 = \frac{k}{\ell}(y - j) + i - i_0.$$

Now the main goal is to calculate the best upper bound for $P_0 = \gcd(z - i_0, y - j_0)$. Starting with

$$\begin{aligned} P_0 &= \gcd\left(\frac{k}{\ell}(y - j) + i - i_0, y - j_0\right) \\ &\leq \gcd(k(y - j) + \ell(i - i_0), k(y - j_0)) = |k(j_0 - j) + \ell(i - i_0)|, \end{aligned}$$

we need to consider $\kappa = |k(j_0 - j) + \ell(i - i_0)|$. Clearly, it is non-zero, further the three cases

$$j \neq j_0,\ i \neq i_0, \qquad j \neq j_0,\ i = i_0, \qquad j = j_0,\ i \neq i_0$$

give the upper bounds $2(k + \ell)$, $2k$, 2ℓ on κ, respectively. Then using inequality (3.1), it yields

$$\begin{aligned} \alpha^{\frac{z-1}{2}} < P = \gcd\left(B_z - \varepsilon_z, B_y - \varepsilon_y\right) &\leq \prod_{i,j\in\{\pm 1\}} B_{\gcd(z-i,y-j)} \\ &\leq \alpha^{\frac{z+1}{k}+2(k+\ell)+2k+2\ell-4\cdot 0.983}. \end{aligned}$$

Going through the eligible pairs

$$(k, \ell) = (3, 2),\ (4, 3),\ (5, 3),\ (5, 4),\ (6, 5),\ (7, 4),\ (7, 5),\ (7, 6),$$

the previous argument provides the upper bounds

$$z < 105.1,\ 101.8,\ 98,\ 111.3,\ 124.1,\ 115.8,\ 127,\ 138.2,$$

respectively. The premise $z \geq 139$ contradicts any of these bounds.

Case 3.2 $z \leq 138$

We ran a computer search to detect all positive integer solutions to system (1.2). Note that the case $\varepsilon_x = \varepsilon_y = \varepsilon_z = 1$ has already been solved in [2]. Observe that

$$a = \sqrt{\frac{(B_x - \varepsilon_x)(B_y - \varepsilon_y)}{B_z - \varepsilon_z}}$$

is integer. Clearly, analogous formulae exist for b and c, too. Taking the range $1 \leq x \leq y < z \leq 138$, and checking if the above numbers a, b and c are all integer, we found only the solution given by (1.3). Thus the proof of the theorem is complete.

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