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Shifted poly-Cauchy numbers*

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Abstract. Recently, the first author introduced the concept of poly-Cauchy numbers as a generalization of the classical Cauchy numbers and an analogue of poly-Bernoulli numbers. This concept has been generalized in various ways, including poly-Cauchy numbers with a q parameter. In this paper, we give a different kind of generalization called shifted poly-Cauchy numbers and investigate several arithmetical properties. Such numbers can be expressed in terms of original poly-Cauchy numbers. This concept is a kind of analogous ideas to that of Hurwitz zeta-functions compared to Riemann zeta-functions.

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1 Introduction

Recently, the first author (see [10]) introduced the poly-Cauchy numbers $c_n^{(k)}$ for a positive integer k and a nonnegative integer n, given by

$$\mathrm{Lif}_k\bigl(\ln(1+x)\bigr) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}, \tag{1.1}$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+1)^k}$$

are the polylogarithm factorial functions. This concept is an analogue of poly-Bernoulli numbers $B_n^{(k)}$ introduced by Kaneko [9], where $B_n^{(k)}$ are defined by

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-x})}{1-\mathrm{e}^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

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where

$$\mathrm{Li}_k(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^k}$$

is the kth polylogarithm function. When $k=1$, $B_n^{(1)}=B_n$ is the classical Bernoulli number with $B_1^{(1)}=1/2$.

When $k=1$, $c_n^{(1)}=c_n$ are the Cauchy numbers (see [5]) defined by

$$c_n=\int\limits_0^1 x(x-1)\cdots(x-n+1)\,\mathrm{d}x.$$

The numbers $c_n/n!$ are sometimes called the Bernoulli numbers of the second kind (see, e.g., [1, 17]). Such numbers have been studied by several authors (see [4, 14, 15, 16, 18]) because they are related to various special combinatorial numbers, including Stirling numbers of both kinds, Bernoulli numbers, and harmonic numbers. The poly-Cauchy numbers $c_n^{(k)}$ are also given by

$$c_n^{(k)}=\underbrace{\int\limits_0^1\cdots\int\limits_0^1}_{k}(x_1x_2\ldots x_k)(x_1x_2\ldots x_k-1)\cdots(x_1x_2\ldots x_k-n+1)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\ldots\,\mathrm{d}x_k.$$

Denote by $\left[{n \atop m}\right]$ the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$, arising as the coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}x^m$$

(see, e.g., [7]). Then, as seen in [10, Thm. 1], the poly-Cauchy numbers $c_n^{(k)}$ can be expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$.

Proposition 1.

$$c_n^{(k)}=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(-1)^{n-m}}{(m+1)^k}\quad(n\geqslant 0,\ k\geqslant 1).$$

As one general case of the poly-Cauchy numbers, the poly-Cauchy numbers with a q parameter $c_{n,q}^{(k)}$ (see [11]) are defined by

$$c_{n,q}^{(k)}=\underbrace{\int\limits_0^1\cdots\int\limits_0^1}_{k}(x_1x_2\ldots x_k)(x_1x_2\ldots x_k-q)\cdots\bigl(x_1x_2\ldots x_k-(n-1)q\bigr)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\ldots\,\mathrm{d}x_k$$

and expressed as

$$c_{n,q}^{(k)}=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(-q)^{n-m}}{(m+1)^k}\quad(n\geqslant 0,\ k\geqslant 1)$$

(see [11, Thm. 1]). In this paper, we give a different kind of generalization of poly-Cauchy numbers.

The Hurwitz zeta-function $\zeta(s,q)=\sum_{n=0}^{\infty}1/(q+n)^s$ is a generalization of the famous Riemann zeta-function $\zeta(s)=\sum_{n=1}^{\infty}1/n^s$ since $\zeta(s)=\zeta(s,1)$. A similar directed extension can be seen in the function in [6] as a natural extension of the Arakawa–Kaneko function, which is closely related to poly-Bernoulli numbers $B_n^{(k)}$ (see [2]). One strong motivation in [6] was to generalize the Arakawa–Kaneko function, which is related to poly-Bernoulli numbers and multiple zeta values (see [2]). Some functions like poly-Cauchy numbers and/or polynomials corresponding to the Arakawa–Kaneko function have been considered too. For instance, Shibukawa and the first author [13] consider the function

$$\hat{\zeta}_\alpha^k(s,z):=\frac{1}{\Gamma(s-\alpha)}\int_0^1 t^{-\alpha-1}(1-t)^{z-1}\bigl(-\log(1-t)\bigr)^s\,\mathrm{Lif}_k\bigl(\ln(1-t)\bigr)\,\mathrm{d}t\quad\bigl(\mathfrak{Re}(s)>\mathfrak{Re}(\alpha)\bigr),$$

yielding $\zeta_{l+1}^k(1,z)=c_l^{(k)}(1-z)$. Kamano and the first author [8] consider the function

$$Z_k(s):=\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\,\mathrm{Lif}_k\bigl(\ln(1-t)\bigr)\,\mathrm{d}t\quad\bigl(\mathfrak{Re}(s)>0\bigr),$$

yielding $\hat{Z}_k(-n)=c_n^{(k)}$ $(n\geqslant 0)$.

Hence, as a different direction of generalization of poly-Cauchy numbers $c_n^{(k)}$, consider the value

$$c_{n,\alpha}^{(k)}=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(-1)^{n-m}}{(m+\alpha)^k}$$

for a positive real number α. For example, if $n=5$ and $\alpha=3$, then

$$c_5^{(k)}=\frac{24}{2^k}-\frac{50}{3^k}+\frac{35}{4^k}-\frac{10}{5^k}+\frac{1}{6^k},\qquad c_{5,3}^{(k)}=\frac{24}{4^k}-\frac{50}{5^k}+\frac{35}{6^k}-\frac{10}{7^k}+\frac{1}{8^k}.$$

In this paper, we give a different kind of generalization called shifted poly-Cauchy numbers, using the generalized polylogarithm factorial function $\mathrm{Lif}_k(z;\alpha)=\sum_{m=0}^{\infty}z^m/m!\,(m+\alpha)^k$, which is relevant to the Hurwitz zeta-function and a general Lerch zeta-function $\Phi(z,s,\alpha)=\sum_{n=0}^{\infty}z^n/(n+\alpha)^s$ (see [3]). We also investigate several arithmetical properties and formulae related with the Stirling numbers of the first and second kind and with some generalizations of Bernoulli numbers.

2 Definitions and basic properties

Let $n\geqslant 0$ and $k\geqslant 1$ be integers, and $\alpha\neq 0$ be a positive real number. Define $c_{n,\alpha}^{(k)}$ by

$$c_{n,\alpha}^{(k)}=\underbrace{\int_0^1\cdots\int_0^1}_{k}(x_1\dots x_k)^\alpha(x_1\dots x_k-1)\cdots(x_1\dots x_k-n+1)\,\mathrm{d}x_1\,\dots\,\mathrm{d}x_k.$$

Then, $c_{n,\alpha}^{(k)}$ can be expressed in terms of the Stirling numbers of the first kind $\left[\begin{smallmatrix}n\\m\end{smallmatrix}\right]$.

Theorem 1. *Let α be a positive real number. Then*

$$c_{n,\alpha}^{(k)}=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(-1)^{n-m}}{(m+\alpha)^k}\quad(n\geqslant 0,\ k\geqslant 1).$$

Remark. Without the definition of integrals, k may be any integer. When $\alpha = 1$, Theorem 1 is reduced to Proposition 1.

Proof. By

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} x^m$$

we have

$$c_{n,\alpha}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} (x_1 \ldots x_k)^{m+\alpha-1} \,\mathrm{d}x_1 \ldots \mathrm{d}x_k = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^{n-m}}{(m+\alpha)^k}. \qquad \square$$

For an integer k and a positive real number α, define the function $\mathrm{Lif}_k(z;\alpha)$ by

$$\mathrm{Lif}_k(z;\alpha) = \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+\alpha)^k}.$$

When $\alpha = 1$, $\mathrm{Lif}_k(z;1) = \mathrm{Lif}_k(z)$ is the *polylogarithm factorial* function (see [10]).

Theorem 2. *The generating function of the number $c_{n,\alpha}^{(k)}$ is given by*

$$\mathrm{Lif}_k\bigl(\ln(1+x);\alpha\bigr) = \sum_{m=0}^{\infty} c_{n,\alpha}^{(k)} \frac{x^n}{n!}.$$

Remark. When $\alpha = 1$, Theorem 2 is reduced to [10, Thm. 2].

Proof. Since

$$\frac{(\ln(1+x))^m}{m!} = (-1)^m \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-x)^n}{n!},$$

by Theorem 1 we have

$$\begin{aligned}\sum_{n=0}^{\infty} c_{n,\alpha}^{(k)} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^{n-m}}{(m+\alpha)^k} \frac{x^n}{n!} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+\alpha)^k} \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-x)^n}{n!} \\ &= \sum_{m=0}^{\infty} \frac{(\ln(1+x))^m}{m!\,(m+\alpha)^k} = \mathrm{Lif}_k\bigl(\ln(1+x);\alpha\bigr). \qquad \square\end{aligned}$$

Remark. The value k is not necessarily positive as seen in the proof. Namely, according to the definition by the integrals, k should be a positive integer. But, after Theorem 1, k can be nonpositive.

The generating function of the number $c_{n,\alpha}^{(k)}$ can be written in the form of iterated integrals.

Corollary 1. *Let α be a positive real number. For $k = 1$, we have*

$$\frac{1}{(\ln(1+x))^\alpha} \int_0^x \bigl(\ln(1+x)\bigr)^{\alpha-1} \,\mathrm{d}x = \sum_{n=0}^{\infty} c_{n,\alpha}^{(1)} \frac{x^n}{n!}.$$

For $k>1$, we have

$$\frac{1}{(\ln(1+x))^{\alpha}}\underbrace{\int\limits_0^x \frac{1}{(1+x)\ln(1+x)}\int\limits_0^x\cdots\frac{1}{(1+x)\ln(1+x)}\int\limits_0^x}_{k}\big(\ln(1+x)\big)^{\alpha-1}\underbrace{\mathrm{d}x\ldots\mathrm{d}x}_{k}$$
$$=\sum_{n=0}^{\infty}c_{n,\alpha}^{(k)}\frac{x^n}{n!}.$$

Remark. When $\alpha=1$, Corollary 1 is reduced to [10, Cor. 1].

Proof. For $k=1$,

$$\begin{aligned}\mathrm{Lif}_1(z;\alpha)&=\sum_{m=0}^{\infty}\frac{z^m}{m!\,(m+\alpha)}=\frac{1}{z^\alpha}\sum_{m=0}^{\infty}\frac{z^{m+\alpha}}{m!\,(m+\alpha)}\\&=\frac{1}{z^\alpha}\int\limits_0^z\sum_{m=0}^{\infty}\frac{z^{m+\alpha-1}}{m!}\,\mathrm{d}z=\frac{1}{z^\alpha}\int\limits_0^z z^{\alpha-1}\mathrm{e}^z\,\mathrm{d}z\\&=\frac{1}{z^\alpha}\left((-1)^\alpha(\alpha-1)!+\mathrm{e}^z\sum_{i=0}^{\alpha-1}(-1)^i\frac{(\alpha-1)!}{(\alpha-i-1)!}z^{\alpha-i-1}\right)\quad\text{(if }\alpha\text{ is an integer)}.\end{aligned}$$

For $k>1$, we have

$$\mathrm{Lif}_k(z;\alpha)=\frac{1}{z^\alpha}\sum_{m=0}^{\infty}\frac{z^{m+\alpha}}{m!\,(m+\alpha)^k}=\frac{1}{z^\alpha}\int\limits_0^z\sum_{m=0}^{\infty}\frac{z^{m+\alpha-1}}{m!\,(m+\alpha)^{k-1}}\,\mathrm{d}z=\frac{1}{z^\alpha}\int\limits_0^z z^{\alpha-1}\,\mathrm{Lif}_{k-1}(z;\alpha)\,\mathrm{d}z.$$

Hence,

$$\mathrm{Lif}_k(z;\alpha)=\frac{1}{z^\alpha}\underbrace{\int\limits_0^z\frac{1}{z}\int\limits_0^z\cdots\frac{1}{z}\int\limits_0^z\frac{1}{z}\int\limits_0^z}_{k}z^{\alpha-1}\mathrm{e}^z\underbrace{\mathrm{d}z\ldots\mathrm{d}z}_{k}.$$

Putting $z=\ln(1+x)$, we get the result. □

The numbers $c_{n,\alpha}^{(k)}$ also have a relation with the Stirling numbers of the second kind $\left\{{n \atop m}\right\}$, determined by

$$\left\{{n\atop m}\right\}=\frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(m-j)^n$$

(see, e.g., [7]).

Theorem 3. *Let k be an integer, and α be a positive real number. Then*

$$\sum_{m=0}^{n}\left\{{n\atop m}\right\}c_{n,\alpha}^{(k)}=\frac{1}{(n+\alpha)^k}.$$

Remark. When $\alpha = 1$, Theorem 3 is reduced to [10, Thm. 3].

Proof. Using the inversion formula

$$\sum_{m=0}^{\max\{l,n\}} (-1)^{m-n} \begin{bmatrix} m \\ l \end{bmatrix} \begin{Bmatrix} n \\ m \end{Bmatrix} = \begin{cases} 1 & (l = n), \\ 0 & (l \neq n) \end{cases}$$

(see [7, Chap. 6]) and Theorem 1, we have

$$\begin{aligned} \sum_{m=0}^{n} \begin{Bmatrix} n \\ m \end{Bmatrix} c_{n,\alpha}^{(k)} &= \sum_{m=0}^{n} \begin{Bmatrix} n \\ m \end{Bmatrix} (-1)^m \sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix} \frac{(-1)^l}{(l+\alpha)^k} = \sum_{l=0}^{n} \frac{(-1)^l}{(l+\alpha)^k} \sum_{m=l}^{n} (-1)^m \begin{bmatrix} m \\ l \end{bmatrix} \begin{Bmatrix} n \\ m \end{Bmatrix} \\ &= \frac{(-1)^n}{(n+\alpha)^k} (-1)^n \cdot 1 = \frac{1}{(n+\alpha)^k}. \qquad \square \end{aligned}$$

3 Shifted poly-Cauchy numbers in terms of original poly-Cauchy numbers

Shifted poly-Cauchy numbers can be expressed in terms of original poly-Cauchy numbers. For example, putting $\alpha = 1, 2, \dots, 6$, we have

$$\begin{aligned} c_{n,1}^{(k)} &= c_n^{(k)}, \\ c_{n,2}^{(k)} &= c_{n+1}^{(k)} + n c_n^{(k)}, \\ c_{n,3}^{(k)} &= c_{n+2}^{(k)} + (2n+1) c_{n+1}^{(k)} + n^2 c_n^{(k)}, \\ c_{n,4}^{(k)} &= c_{n+3}^{(k)} + 3(n+1) c_{n+2}^{(k)} + \bigl(3n^2 + 3n + 1\bigr) c_{n+1}^{(k)} + n^3 c_n^{(k)}, \\ c_{n,5}^{(k)} &= c_{n+4}^{(k)} + (4n+6) c_{n+3}^{(k)} + \bigl(6n^2 + 12n + 7\bigr) c_{n+2}^{(k)} + \bigl(4n^3 + 6n^2 + 4n + 1\bigr) c_{n+1}^{(k)} + n^4 c_n^{(k)}, \\ c_{n,6}^{(k)} &= c_{n+5}^{(k)} + 5(n+2) c_{n+4}^{(k)} + 5\bigl(2n^2 + 6n + 5\bigr) c_{n+3}^{(k)} + 5\bigl(2n^3 + 6n^2 + 7n + 3\bigr) c_{n+2}^{(k)} \\ &\quad + \bigl(5n^4 + 10n^3 + 10n^2 + 5n + 1\bigr) c_{n+1}^{(k)} + n^5 c_n^{(k)}. \end{aligned}$$

In general, we can state the following relation.

Theorem 4. *For a positive integer α, we have*

$$c_{n,\alpha}^{(k)} = \sum_{\mu=0}^{\alpha-1} Q_\mu(n,\alpha) c_{n+\mu}^{(k)} \quad (n \geqslant 0),$$

where

$$Q_\mu(n,\alpha) = \sum_{i=0}^{\alpha-\mu-1} \binom{\alpha-1}{i} \begin{Bmatrix} \alpha-i-1 \\ \mu \end{Bmatrix} n^i \quad (0 \leqslant \mu \leqslant \alpha-1).$$

We need the following lemma in order to prove Theorem 4. Let α be a positive integer.

Lemma 1.

$$\sum_{\mu=0}^{\alpha-1} (-1)^\mu Q_{\alpha-\mu-1}(n,\alpha) \begin{bmatrix} n+\alpha-\mu-1 \\ n+\alpha-m-1 \end{bmatrix} = \begin{cases} \left[{n \atop n-m}\right] & \textit{if } m = 0, 1, \dots, n-1, \\ 0 & \textit{if } m = n, n+1, \dots, \alpha+n-2. \end{cases}$$

Proof. By the definition, if $m > n$ or $m = n \neq 0$, then

$$\begin{bmatrix} n \\ n-m \end{bmatrix} = 0.$$

Put

$$f(\alpha) = \sum_{\mu=0}^{\alpha-1} (-1)^{\mu} Q_{\alpha-\mu-1}(n,\alpha) \begin{bmatrix} n+\alpha-\mu-1 \\ n+\alpha-m-1 \end{bmatrix}.$$

Notice that $Q_{\alpha-1}(n,\alpha) = 1$ and $Q_0(n,\alpha) = n^{\alpha-1}$. When $\alpha = 1$,

$$f(1) = Q_0(n,1) \begin{bmatrix} n \\ n-m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix}.$$

By

$$\mu \begin{Bmatrix} \alpha-i-2 \\ \mu \end{Bmatrix} + \begin{Bmatrix} \alpha-i-2 \\ \mu-1 \end{Bmatrix} = \begin{Bmatrix} \alpha-i-1 \\ \mu \end{Bmatrix}$$

and

$$\binom{\alpha-1}{i} = \binom{\alpha-2}{i} + \binom{\alpha-2}{i-1}$$

we have

$$\begin{aligned}
&(n+\mu) Q_{\mu}(n,\alpha-1) + Q_{\mu-1}(n,\alpha-1) \\
&= (n+\mu) \sum_{i=0}^{\alpha-\mu-2} \binom{\alpha-2}{i} \begin{Bmatrix} \alpha-i-2 \\ \mu \end{Bmatrix} n^i + \sum_{i=0}^{\alpha-\mu-1} \binom{\alpha-2}{i} \begin{Bmatrix} \alpha-i-2 \\ \mu-1 \end{Bmatrix} n^i \\
&= \sum_{i=0}^{\alpha-\mu-2} \binom{\alpha-2}{i} \begin{Bmatrix} \alpha-i-2 \\ \mu \end{Bmatrix} n^{i+1} + \sum_{i=0}^{\alpha-\mu-1} \binom{\alpha-2}{i} \begin{Bmatrix} \alpha-i-1 \\ \mu \end{Bmatrix} n^i \\
&= \sum_{i=1}^{\alpha-\mu-1} \binom{\alpha-2}{i-1} \begin{Bmatrix} \alpha-i-1 \\ \mu \end{Bmatrix} n^i + \sum_{i=0}^{\alpha-\mu-1} \binom{\alpha-2}{i} \begin{Bmatrix} \alpha-i-1 \\ \mu \end{Bmatrix} n^i \\
&= \sum_{i=0}^{\alpha-\mu-1} \binom{\alpha-1}{i} \begin{Bmatrix} \alpha-i-1 \\ \mu \end{Bmatrix} n^i = Q_{\mu}(n,\alpha).
\end{aligned}$$

Therefore, putting $\mu = \alpha-2, \alpha-3, \dots, 2, 1$ in

$$Q_{\mu}(n,\alpha) = (n+\mu) Q_{\mu}(n,\alpha-1) + Q_{\mu-1}(n,\alpha-1),$$

for $\alpha > 1$, we obtain

$$\begin{aligned}
f(\alpha) &= Q_{\alpha-1}(n,\alpha) \begin{bmatrix} n+\alpha-1 \\ n-m+\alpha-1 \end{bmatrix} - Q_{\alpha-2}(n,\alpha) \begin{bmatrix} n+\alpha-2 \\ n-m+\alpha-1 \end{bmatrix} \\
&\quad + Q_{\alpha-3}(n,\alpha) \begin{bmatrix} n+\alpha-3 \\ n-m+\alpha-1 \end{bmatrix} - \cdots
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{\alpha-2}Q_1(n,\alpha)\begin{bmatrix} n+1 \\ n-m+\alpha-1 \end{bmatrix}+(-1)^{\alpha-1}Q_0(n,\alpha)\begin{bmatrix} n \\ n-m+\alpha-1 \end{bmatrix}\\
=&\begin{bmatrix} n+\alpha-1 \\ n-m+\alpha-1 \end{bmatrix}-(n+\alpha-2)\begin{bmatrix} n+\alpha-2 \\ n-m+\alpha-1 \end{bmatrix}\\
&-Q_{\alpha-3}(n,\alpha-1)\left(\begin{bmatrix} n+\alpha-2 \\ n-m+\alpha-1 \end{bmatrix}-(n+\alpha-3)\begin{bmatrix} n+\alpha-3 \\ n-m+\alpha-1 \end{bmatrix}\right)\\
&+Q_{\alpha-4}(n,\alpha-1)\left(\begin{bmatrix} n+\alpha-3 \\ n-m+\alpha-1 \end{bmatrix}-(n+\alpha-4)\begin{bmatrix} n+\alpha-4 \\ n-m+\alpha-1 \end{bmatrix}\right)-\cdots\\
&+(-1)^{\alpha-2}Q_0(n,\alpha-1)\left(\begin{bmatrix} n+1 \\ n-m+\alpha-1 \end{bmatrix}-n\begin{bmatrix} n \\ n-m+\alpha-1 \end{bmatrix}\right)\\
=&\,Q_{\alpha-2}(n,\alpha-1)\begin{bmatrix} n+\alpha-2 \\ n-m+\alpha-2 \end{bmatrix}-Q_{\alpha-3}(n,\alpha-1)\begin{bmatrix} n+\alpha-3 \\ n-m+\alpha-2 \end{bmatrix}\\
&+Q_{\alpha-4}(n,\alpha-1)\begin{bmatrix} n+\alpha-4 \\ n-m+\alpha-2 \end{bmatrix}-\cdots\\
&+(-1)^{\alpha-3}Q_1(n,\alpha-1)\begin{bmatrix} n+1 \\ n-m+\alpha-2 \end{bmatrix}+(-1)^{\alpha-2}Q_0(n,\alpha-1)\begin{bmatrix} n \\ n-m+\alpha-2 \end{bmatrix}\\
=&\,f(\alpha-1). \qquad \square
\end{aligned}
$$

Proof of Theorem 4. For simplicity, we write $Q_\mu = Q_\mu(n,\alpha)$ for fixed integers n and α. By Lemma 1 and the equalities $\left[{n \atop k}\right] = 0$ $(n < k)$ and $\left[{n \atop 0}\right] = 0$ $(n > 0)$ we have

$$
\begin{aligned}
\sum_{\mu=0}^{\alpha-1} Q_\mu c_{n+\mu}^{(k)} &= \sum_{\mu=0}^{\alpha-1} Q_\mu \sum_{m=0}^{n+\mu}\begin{bmatrix} n+\mu \\ m \end{bmatrix}\frac{(-1)^{n+\mu-m}}{(m+1)^k}\\
&= \sum_{\mu=0}^{\alpha-1}(-1)^{\alpha-\mu-1}Q_\mu \sum_{m=0}^{n+\alpha-1}\begin{bmatrix} n+\mu \\ m \end{bmatrix}\frac{(-1)^{\alpha+n-1-m}}{(m+1)^k}\\
&= \sum_{\mu=0}^{\alpha-1}(-1)^{\alpha-\mu-1}Q_\mu \sum_{m=0}^{n+\alpha-1}\begin{bmatrix} n+\mu \\ n+\alpha-m-1 \end{bmatrix}\frac{(-1)^{m}}{(n-m+\alpha)^k}\\
&= \sum_{\mu=0}^{\alpha-1}(-1)^{\mu}Q_{\alpha-\mu-1} \sum_{m=0}^{n+\alpha-2}\begin{bmatrix} n+\alpha-\mu-1 \\ n+\alpha-m-1 \end{bmatrix}\frac{(-1)^{m}}{(n-m+\alpha)^k}\\
&= \sum_{m=0}^{n-1}\sum_{\mu=0}^{\alpha-1}(-1)^{\mu}Q_{\alpha-\mu-1}\begin{bmatrix} n+\alpha-\mu-1 \\ n+\alpha-m-1 \end{bmatrix}\frac{(-1)^{m}}{(n-m+\alpha)^k}\\
&\quad+\sum_{m=n}^{n+\alpha-2}\sum_{\mu=0}^{\alpha-1}(-1)^{\mu}Q_{\alpha-\mu-1}\begin{bmatrix} n+\alpha-\mu-1 \\ n+\alpha-m-1 \end{bmatrix}\frac{(-1)^{m}}{(n-m+\alpha)^k}\\
&= \sum_{m=0}^{n}\begin{bmatrix} n \\ n-m \end{bmatrix}\frac{(-1)^{m}}{(n-m+\alpha)^k} = c_{n,\alpha}^{(k)}.
\end{aligned}
$$

Hence, the proof is done. $\square$

Remark. We can write as

$$Q_\mu(n,\alpha)=\sum_{i=0}^{\alpha-\mu-1}\left\{{\alpha-1\atop \mu+i}\right\}\binom{\mu+i}{\mu}\frac{n!}{(n-i)!}$$

since

$$\begin{aligned}Q_\mu(n,\alpha)&=\sum_{i=0}^{\alpha-\mu-1}\left\{{\alpha-1\atop \mu+i}\right\}\binom{\mu+i}{\mu}\frac{n!}{(n-i)!}=\sum_{i=0}^{\alpha-\mu-1}\left\{{\alpha-1\atop \mu+i}\right\}\binom{\mu+i}{\mu}\sum_{\nu=0}^{i}(-1)^{i-\nu}\left[{i\atop \nu}\right]n^\nu\\&=\sum_{\nu=0}^{\alpha-\mu-1}n^\nu\sum_{i=\nu}^{\alpha-\mu-1}\left\{{\alpha-1\atop \mu+i}\right\}\binom{\mu+i}{\mu}(-1)^{i-\nu}\left[{i\atop \nu}\right]=\sum_{\nu=0}^{\alpha-\mu-1}n^\nu\binom{\alpha-1}{\nu}\left\{{\alpha-\nu-1\atop \mu}\right\}.\end{aligned}$$

Notice that

$$\sum_{i=\nu}^{\alpha-\mu-1}\left\{{\alpha-1\atop \mu+i}\right\}\binom{\mu+i}{\mu}(-1)^{i-\nu}\left[{i\atop \nu}\right]=\binom{\alpha-1}{\nu}\left\{{\alpha-\nu-1\atop \mu}\right\}.$$

4 Poly-Cauchy numbers of the second kind

In [10], the concept of poly-Cauchy numbers of the second kind is also introduced. The poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)}=\underbrace{\int_0^1\dots\int_0^1}_{k}(-x_1x_2\dots x_k)(-x_1x_2\dots x_k-1)\cdots(-x_1x_2\dots x_k-n+1)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\dots\,\mathrm{d}x_k,$$

and the generating function is given by

$$\mathrm{Lif}_k\big(-\ln(1+x)\big)=\sum_{n=0}^{\infty}\hat{c}_n^{(k)}\frac{x^n}{n!}.$$

Then, the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ can be also expressed in terms of the Stirling numbers of the first kind (see [10, Thm. 4]).

Proposition 2.

$$\hat{c}_n^{(k)}=(-1)^n\sum_{m=0}^{n}\left[{n\atop m}\right]\frac{1}{(m+1)^k}.$$

Let α be a positive real number. Similarly to the shifted poly-Cauchy numbers of the first kind $c_{n,\alpha}^{(k)}$, define the shifted poly-Cauchy numbers of the second kind $\hat{c}_{n,\alpha}^{(k)}$ ($n\geqslant 0$, $k\geqslant 1$) by

$$\hat{c}_{n,\alpha}^{(k)}=(-1)^{\alpha-1}\underbrace{\int_0^1\dots\int_0^1}_{k}(-x_1\dots x_k)^\alpha(-x_1\dots x_k-1)\cdots(-x_1\dots x_k-n+1)\,\mathrm{d}x_1\,\dots\,\mathrm{d}x_k.$$

Then, similarly to Theorem 1, $\hat{c}_{n,\alpha}^{(k)}$ can be also expressed in terms of the Stirling numbers of the first kind as a generalization of Proposition 2.

Theorem 5.

$$\hat{c}_{n,\alpha}^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+\alpha)^k} \quad (n \geqslant 0,\ k \geqslant 1).$$

Theorem 6. *The generating function of the number $\hat{c}_{n,\alpha}^{(k)}$ is given by*

$$\mathrm{Lif}_k\bigl(-\ln(1+x);\alpha\bigr) = \sum_{m=0}^{\infty} \hat{c}_{n,\alpha}^{(k)} \frac{x^n}{n!},$$

where

$$\mathrm{Lif}_k(z;\alpha) = \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+\alpha)^k}.$$

Remark. When $\alpha = 1$, Theorem 6 is reduced to [10, Thm. 5].

The generating function of the number $\hat{c}_{n,\alpha}^{(k)}$ can be written in the form of iterated integrals.

Corollary 2. *Let α be a positive real number. For $k = 1$, we have*

$$\frac{1}{(\ln(1+x))^\alpha} \int_0^x \frac{(\ln(1+x))^{\alpha-1}}{(1+x)^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \hat{c}_{n,\alpha}^{(1)} \frac{x^n}{n!}.$$

For $k > 1$, we have

$$\frac{1}{(\ln(1+x))^\alpha} \underbrace{\int_0^x \frac{1}{(1+x)\ln(1+x)} \int_0^x \cdots \frac{1}{(1+x)\ln(1+x)} \int_0^x}_{k} \frac{(\ln(1+x))^{\alpha-1}}{(1+x)^2} \underbrace{\mathrm{d}x \ldots \mathrm{d}x}_{k}$$
$$= \sum_{n=0}^{\infty} \hat{c}_{n,\alpha}^{(k)} \frac{x^n}{n!}.$$

Remark. When $\alpha = 1$ in the first identity, where $k = 1$, we have the generating function of the classical Cauchy numbers of the second kind:

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \hat{c}_n \frac{x^n}{n!}.$$

When $\alpha = 1$, the second identity is reduced to that of Corollary 2 in [10].

The number $\hat{c}_{n,\alpha}^{(k)}$ also has a relation with the Stirling numbers of the second kind.

Theorem 7. *Let k be an integer, and α be a positive real number. Then*

$$\sum_{m=0}^{n} \begin{Bmatrix} n \\ m \end{Bmatrix} \hat{c}_{n,\alpha}^{(k)} = \frac{(-1)^n}{(n+\alpha)^k}.$$

Remark. When $\alpha = 1$, Theorem 7 is reduced to [10, Thm. 6].

In addition, there are relations between both kinds of poly-Cauchy numbers.

Theorem 8. *Let k be an integer, and α be a positive real number. For $n \geqslant 1$, we have*

$$(-1)^n \frac{c_{n,\alpha}^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\hat{c}_{m,\alpha}^{(k)}}{m!}, \qquad (-1)^n \frac{\hat{c}_{n,\alpha}^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{c_{m,\alpha}^{(k)}}{m!}.$$

Remark. When $\alpha = 1$, Theorem 8 is reduced to [10, Thm. 7].

Proof. We shall prove the first identity. The second one is proven similarly and omitted. Using the identity (see, e.g., [7, Chap. 6])

$$\frac{(-1)^l}{n!} \begin{bmatrix} n \\ l \end{bmatrix} = \sum_{m=l}^{n} \frac{(-1)^m}{m!} \binom{n-1}{m-1} \begin{bmatrix} m \\ l \end{bmatrix}$$

and Theorems 1 and 5, we have

$$\begin{aligned} \text{RHS} &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{(-1)^m}{m!} \sum_{l=1}^{m} \begin{bmatrix} m \\ l \end{bmatrix} \frac{1}{(l+\alpha)^k} = \sum_{l=1}^{n} \frac{1}{(l+\alpha)^k} \sum_{m=l}^{n} \frac{(-1)^m}{m!} \binom{n-1}{m-1} \begin{bmatrix} m \\ l \end{bmatrix} \\ &= \sum_{l=1}^{n} \frac{1}{(l+\alpha)^k} \frac{(-1)^l}{n!} \begin{bmatrix} n \\ l \end{bmatrix} = \text{LHS}. \qquad \square \end{aligned}$$

Finally, similarly to Theorem 4, the shifted poly-Cauchy numbers of the second kind can be expressed in terms of the original poly-Cauchy numbers of the second kind.

Theorem 9. *Let α be a positive integer. Then*

$$\hat{c}_{n,\alpha}^{(k)} = (-1)^{\alpha-1} \sum_{\mu=0}^{\alpha-1} Q_\mu(n,\alpha) \hat{c}_{n+\mu}^{(k)} \quad (n \geqslant 0),$$

where $Q_\mu(n,\alpha)$ are the same as in Theorem 4.

5 Some expressions of poly-Cauchy numbers with negative indices

It is known that the poly-Bernoulli numbers satisfy the duality theorem $B_n^{(-k)} = B_k^{(-n)}$ for $n, k \geqslant 0$ (see [9, Thm. 2]) because of the symmetric formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{\mathrm{e}^{x+y}}{\mathrm{e}^x + \mathrm{e}^y - \mathrm{e}^{x+y}}.$$

However, the corresponding duality theorem does not hold for poly-Cauchy numbers for any real number α, as the following results show.

Proposition 3. *For nonnegative integers n and k and a real number $\alpha \neq 0$, we have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,\alpha}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \mathrm{e}^{\alpha y} (1+x)^{\mathrm{e}^y}, \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{c}_{n,\alpha}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{\mathrm{e}^{\alpha y}}{(1+x)^{\mathrm{e}^y}}.$$

Proof. We shall prove the first identity. The second identity is proven similarly. By Theorem 2 we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,\alpha}^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!} &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} c_{n,\alpha}^{(-k)}\frac{x^n}{n!}\right)\frac{y^k}{k!} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(m+\alpha)^k}{m!}\bigl(\ln(1+x)\bigr)^m\frac{y^k}{k!} \\
&= \sum_{m=0}^{\infty}\frac{(\ln(1+x))^m}{m!}\sum_{k=0}^{\infty}\frac{((m+\alpha)y)^k}{k!} \\
&= \sum_{m=0}^{\infty}\frac{(\ln(1+x))^m}{m!}\mathrm{e}^{(m+\alpha)y} = \mathrm{e}^{\alpha y}\sum_{m=0}^{\infty}\frac{(\mathrm{e}^y\ln(1+x))^m}{m!} \\
&= \mathrm{e}^{\alpha y}(1+x)^{\mathrm{e}^y}. \qquad \square
\end{aligned}
$$

By using Proposition 3 we have explicit expressions of the poly-Cauchy numbers with negative indices.

Theorem 10. *For nonnegative integers n, k and a real number $\alpha \neq 0$, we have*

$$
\begin{aligned}
c_{n,\alpha}^{(-k)} &= \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{n+j}j!\left(\begin{bmatrix} n \\ j \end{bmatrix} - n\begin{bmatrix} n-1 \\ j \end{bmatrix}\right)\binom{k}{i}\left\{\begin{matrix} i \\ j \end{matrix}\right\}\alpha^{k-i}, \\
\hat{c}_{n,\alpha}^{(-k)} &= \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{n}j!\begin{bmatrix} n+1 \\ j+1 \end{bmatrix}\binom{k}{i}\left\{\begin{matrix} i \\ j \end{matrix}\right\}\alpha^{k-i}.
\end{aligned}
$$

Remark. If $\alpha = 1$, by

$$
\sum_{i=0}^{k}\binom{k}{i}\left\{\begin{matrix} i \\ j \end{matrix}\right\} = \left\{\begin{matrix} k+1 \\ j+1 \end{matrix}\right\}
$$

(see [7]) the above identities become

$$
\begin{aligned}
c_{n}^{(-k)} &= \sum_{j=0}^{k}(-1)^{n+j}j!\left(\begin{bmatrix} n \\ j \end{bmatrix} - n\begin{bmatrix} n-1 \\ j \end{bmatrix}\right)\left\{\begin{matrix} k+1 \\ j+1 \end{matrix}\right\}, \\
\hat{c}_{n}^{(-k)} &= \sum_{j=0}^{k}(-1)^{n}j!\begin{bmatrix} n+1 \\ j+1 \end{bmatrix}\left\{\begin{matrix} k+1 \\ j+1 \end{matrix}\right\}.
\end{aligned}
$$

Proof. By Proposition 3, together with

$$
\frac{(\mathrm{e}^y-1)^j}{j!} = \sum_{k=j}^{\infty}\left\{\begin{matrix} k \\ j \end{matrix}\right\}\frac{y^k}{k!} \quad \text{and} \quad \frac{(-\ln(1+x))^j}{j!} = \sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-x)^n}{n!}
$$

(see [7]), we have

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,\alpha}^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!} = (1+x)^{\mathrm{e}^y-1}(1+x)\mathrm{e}^{\alpha y} = \exp\bigl(\bigl(\mathrm{e}^y-1\bigr)\ln(1+x)\bigr)(1+x)\mathrm{e}^{\alpha y}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} j! \frac{(\mathrm{e}^y - 1)^j}{j!} \frac{(\ln(1+x))^j}{j!} (1+x)\mathrm{e}^{\alpha y} \\
&= \sum_{j=0}^{\infty} (-1)^j j!\, \mathrm{e}^{\alpha y} \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} (1+x) \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathrm{e}^{\alpha y} \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} &= \sum_{l=0}^{\infty} \frac{(\alpha y)^l}{l!} \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \frac{\alpha^{k-i}}{(k-i)!} \left\{ {i \atop j} \right\} \frac{1}{i!} \right) y^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} \left\{ {i \atop j} \right\} \alpha^{k-i} \right) \frac{y^k}{k!}
\end{aligned}
$$

and

$$
\begin{aligned}
(1+x) \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} &= \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} - \sum_{n=j+1}^{\infty} \left[{n-1 \atop j} \right] \frac{(-1)^n}{(n-1)!} x^n \\
&= \sum_{n=0}^{\infty} \left(\left[{n \atop j} \right] - n \left[{n-1 \atop j} \right] \right) (-1)^n \frac{x^n}{n!},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,\alpha}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} &= \sum_{j=0}^{\infty} (-1)^j j! \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} \left\{ {i \atop j} \right\} \alpha^{k-i} \right) \frac{y^k}{k!} \sum_{n=0}^{\infty} \left(\left[{n \atop j} \right] - n \left[{n-1 \atop j} \right] \right) (-1)^n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{n+j} j! \left(\left[{n \atop j} \right] - n \left[{n-1 \atop j} \right] \right) \binom{k}{i} \left\{ {i \atop j} \right\} \alpha^{k-i} \frac{x^n}{n!} \frac{y^k}{k!}.
\end{aligned}
$$

Similarly, by

$$
\frac{1}{1+x} \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \left[{n+1 \atop j+1} \right] \frac{x^n}{n!}
$$

we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{c}_{n,\alpha}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} &= \frac{\mathrm{e}^{\alpha y}}{(1+x)^{\mathrm{e}^y}} = \exp\bigl(-\bigl(\mathrm{e}^y - 1\bigr) \ln(1+x)\bigr) \frac{\mathrm{e}^{\alpha y}}{1+x} \\
&= \sum_{j=0}^{\infty} j! \frac{(\mathrm{e}^y - 1)^j}{j!} \frac{(-\ln(1+x))^j}{j!} \frac{\mathrm{e}^{\alpha y}}{1+x} \\
&= \sum_{j=0}^{\infty} j!\, \mathrm{e}^{\alpha y} \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} \frac{1}{1+x} \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!}
\end{aligned}
$$

$$= \sum_{j=0}^{\infty} j! \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} \left\{ {i \atop j} \right\} \alpha^{k-i} \right) \frac{y^k}{k!} \sum_{n=0}^{\infty} (-1)^n \left[{n+1 \atop j+1} \right] \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^n j! \left[{n+1 \atop j+1} \right] \binom{k}{i} \left\{ {i \atop j} \right\} \alpha^{k-i} \frac{x^n}{n!} \frac{y^k}{k!}. \qquad \square$$

6 Poly-Cauchy numbers and poly-Bernoulli numbers

In this section, let k be an integer, and α be a positive real number. An explicit form of a poly-Bernoulli number $B_n^{(k)}$ is given by

$$B_n^{(k)} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^{n-m} m!}{(m+1)^k}$$

(see [9, Thm. 1]. In [10, Thm. 8], the following expression of $B_n^{(k)}$ in terms of poly-Cauchy numbers $c_n^{(k)}$ is given.

Proposition 4.

$$B_n^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_l^{(k)} \quad (n \geqslant 1).$$

On the contrary, in [12, Thm. 2.2], another expression of $c_n^{(k)}$ in terms of $B_n^{(k)}$ is given.

Proposition 5.

$$c_n^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \left[{n \atop m} \right] \left[{m \atop l} \right] B_l^{(k)} \quad (n \geqslant 1).$$

We generalize such results by introducing the shifted poly-Bernoulli numbers defined by

$$B_{n,\alpha}^{(k)} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^{n-m} m!}{(m+\alpha)^k} \quad (n \geqslant 0).$$

If $\alpha = 1$, then our results are reduced to the previous ones.

Theorem 11. *For $n \geqslant 0$, we have*

$$B_{n,\alpha}^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_{l,\alpha}^{(k)}, \qquad c_{n,\alpha}^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \left[{n \atop m} \right] \left[{m \atop l} \right] B_{l,\alpha}^{(k)}.$$

Proof. For the first identity,

$$\text{RHS} = \sum_{l=1}^{n} \sum_{m=l}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} (-1)^l \sum_{i=0}^{l} \left[{l \atop i} \right] \frac{(-1)^i}{(i+\alpha)^k}$$

$$= \sum_{i=1}^{n} \frac{(-1)^i}{(i+\alpha)^k} \sum_{l=i}^{n} \sum_{m=l}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} (-1)^l \left[{l \atop i} \right]$$

$$
\begin{aligned}
&=\sum_{i=1}^n\frac{(-1)^i}{(i+\alpha)^k}\sum_{m=i}^n m!\left\{ n \atop m \right\}\sum_{l=i}^m(-1)^l\left\{ m-1 \atop l-1 \right\}\left[l \atop i \right]\\
&=\sum_{i=1}^n\frac{(-1)^i}{(i+\alpha)^k}\sum_{m=i}^n m!\left\{ n \atop m \right\}(-1)^m\binom{m-1}{i-1}\\
&=\sum_{i=1}^n\frac{(-1)^i}{(i+\alpha)^k}(-1)^n i!\left\{ n \atop i \right\}=\text{LHS}.
\end{aligned}
$$

For the second identity,

$$
\begin{aligned}
\text{RHS}&=(-1)^n\sum_{l=1}^n\sum_{m=1}^n\frac{(-1)^m}{m!}\left[n \atop m \right]\left[m \atop l \right](-1)^l\sum_{i=0}^l\left\{ l \atop i \right\}\frac{(-1)^i i!}{(i+\alpha)^k}\\
&=(-1)^n\sum_{m=1}^n\frac{(-1)^m}{m!}\left[n \atop m \right]\sum_{l=0}^n\left[m \atop l \right](-1)^l\sum_{i=0}^l\left\{ l \atop i \right\}\frac{(-1)^i i!}{(i+\alpha)^k}\\
&=(-1)^n\sum_{m=1}^n\frac{(-1)^m}{m!}\left[n \atop m \right]\sum_{i=0}^n\frac{(-1)^i i!}{(i+\alpha)^k}\sum_{l=i}^n(-1)^l\left[m \atop l \right]\left\{ l \atop i \right\}\\
&=(-1)^n\sum_{m=0}^n\frac{(-1)^m}{m!}\left[n \atop m \right]\frac{(-1)^m m!}{(m+\alpha)^k}(-1)^m\\
&=(-1)^n\sum_{m=0}^n\left[n \atop m \right]\frac{(-1)^m}{(m+\alpha)^k}=\text{LHS}.
\end{aligned}
$$

Note that $\left[m \atop 0 \right]=0$ $(m\geqslant 1)$, $\left[m \atop l \right]=0$ $(l>m)$, and

$$
\sum_{l=i}^m(-1)^{m-l}\left[m \atop l \right]\left\{ l \atop i \right\}=\begin{cases}1 & (i=m),\\ 0 & (i\neq m).\end{cases}\qquad\square
$$

Similarly, concerning

$$
\hat{c}_{n,\alpha}^{(k)}=(-1)^n\sum_{m=0}^n\left[n \atop m \right]\frac{1}{(m+\alpha)^k}\quad(n\geqslant 0)
$$

as a generalization of the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$, we have the following:

Theorem 12.

$$
B_{n,\alpha}^{(k)}=(-1)^n\sum_{l=1}^n\sum_{m=1}^n m!\left\{ n \atop m \right\}\left\{ m \atop l \right\}\hat{c}_{l,\alpha}^{(k)},\qquad \hat{c}_{n,\alpha}^{(k)}=(-1)^n\sum_{l=1}^n\sum_{m=1}^n\frac{1}{m!}\left[n \atop m \right]\left[m \atop l \right]B_{l,\alpha}^{(k)}.
$$

Remark. If $\alpha=1$, these results are reduced to the identities in Theorem 3.2 and Theorem 3.1 in [12], respectively.

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